1. A derangement of a string of distinct elements is a rearrangement of the string such that no element appears in its original position. For example, BCA is a derangement of ABC. $D_n$ represents the number of derangements of any string composed of $n$ distinct elements. $D_2 = 1$ and $D_3 = 2$.

(a) What are $D_4$ and $D_5$?
(b) How many derangements are there of the string ABCDEFG?
(c) Find a recursive relationship for $D_n$ in terms of the previous two terms ($D_{n-1}$ and $D_{n-2}$).
(d) Find a recursive relationship for $D_n$ in terms of only the previous term, $D_{n-1}$.

2. Find the number of 3-letter "words" that use letters from the 10-letter set \{A, B, C, ..., J\} in which all letters are different and the letters appear in alphabetical order.

3. Assume that a hand of thirteen cards is dealt from a randomized deck of 52 cards. Let $A$ be the probability that the hand contains two aces. Let $B$ be the probability that it contains two aces if you already know it contains at least one ace. Let $C$ be the probability that it contains at least two aces if you already know it contains an ace of hearts. Write down the inequality relationship between $A, B$ and $C$, i.e. one possibility is $A = B > C$.

4. Find the number of rearrangements of 12345 (including 12345) such that none of the following is true: 1 is in position 5, 2 is in position 1, 3 is in position 2, 4 is in position 4, and 5 is in position 3.

5. Assume that nobody has a birthday on February 29th. How large must a group be so that there is a greater than 50% chance that at least 2 members have the same birthday?

6. Find the number of combinations of length $k$ that use elements from a set of $n$ distinct elements, allowing repetition.

7. Find the number of combinations of length $k$ that use elements from a given set of $n$ distinct elements, allowing repetition and with no missing elements. (Obviously, $k$ must be greater than $n$)

8. An election takes place between two candidates. Candidate A wins by a vote of 1032 to 971. If the votes are counted one at a time and in a random order, determine the probability that the winner was never behind at any point in the counting.

9. Evaluate the sum $\left( \binom{100}{0} + \frac{1}{2} \binom{100}{1} + \frac{1}{3} \binom{100}{2} + \ldots + \frac{1}{101} \binom{100}{100} \right)$.

10. Find the number of distributions of a given set of $m$ identical balls into a given set of $n$ distinct boxes.

11. Find the number of distributions of a given set of $m$ distinct balls into a given set of $n$ distinct boxes if each box must contain a specific number of balls ($m_i$ is the number of balls to be put into box $i$). Please state the answer with only factorials (not in combinatorial notation).
12. Find the coefficient of $X^2Y^3$ in each of the following.
   (a) $(X + Y + 1)^7$
   (b) $(X^2 + Y - 1)^7$

13. Find the number of "words" of length $m$ from a set of $n$ letters, if each letter must occur at least once in each word.

14. Find the number of ways to distribute seven distinct balls into three distinct boxes if each box must contain a different number of balls, allowing an empty box.

15. How many ways can a class of 10 students be divided into two groups of 3 and 1 group of 4?

16. Find the number of subsets $A$ of the set of digits $\{0, 1, 2, 3, \ldots, 9\}$ such that $A$ contains no two consecutive digits. Hint: Find a better statement of the problem; find a recursive formula, and then attempt to solve the problem for the number of digits given.

17. If we are trying to find the number of words of length $m$ from a given set of $n$ letters, with each letter occurring at least once in each word, let us call the answer $T(m, n)$. This is equivalent to finding the number of distribution of a set of $m$ distinct balls into a set of $n$ distinct boxes, if no boxes can be empty. $T(m, n)$ is the sum of all possible partitions of the balls (i.e. we sum all possible ways of putting the balls into boxes (4 in box 1, 2 in box 2, 1 in box 3 for example)). More precisely, if we call $m_i$ to be the number of balls in box $i$, then $T(m, n) = \sum_{m_1 + m_2 + \cdots + m_r = m}^{\text{partition of } m} \frac{m!}{m_1!m_2!\cdots m_r!}$. For example, $T(3, 2) = \frac{3^2}{2!} + \frac{3^1}{3!} = 3 + \frac{3}{6} = 3 + 0.5 = 3.5$. Find a recursive pattern for $T(m, n)$ in terms of previous terms (previous meaning a smaller $m$, a smaller $n$, or both). Hint: set up a sort of "Pascal's Triangle" for $T(m, n)$. Prove your answer using words.

18. You have an infinite number of 1 cent, 2 cent, and 5 cent stamps. You are trying to post a letter that requires $n$ cents of postage stamps, where $n > 8$. Let $a(n)$ be the number of sequences of stamps that give exactly the required postage of $n$ cents (i.e. order matters). Find $a(n)$ in terms of previous terms of the sequence of $a$'s, using as few previous terms as possible.

19. Suppose we have $n$ lines in a plane in general position, which means that none are parallel to each other and that no three of these lines intersect at a single point. Find the number of regions that these lines divide the plane into...
   (a) in a recursive form.
   (b) in a nonrecursive formula.

20. Find the 2000th positive integer that is not the difference between any two integer squares.