Harvard-MIT Mathematics Tournament  
March 15, 2003  

Individual Round: Algebra Subject Test  

1. Find the smallest value of \( x \) such that \( a \geq 14\sqrt{a} - x \) for all nonnegative \( a \).
2. Compute \( \frac{\tan^2(20^\circ) - \sin^2(20^\circ)}{\tan^2(20^\circ) \sin^2(20^\circ)} \).
3. Find the smallest \( n \) such that \( n! \) ends in 290 zeroes.
4. Simplify: \( 2\sqrt{1.5 + \sqrt{2}} - (1.5 + \sqrt{2}) \).
5. Several positive integers are given, not necessarily all different. Their sum is 2003. Suppose that \( n_1 \) of the given numbers are equal to 1, \( n_2 \) of them are equal to 2, \ldots, \( n_{2003} \) of them are equal to 2003. Find the largest possible value of \( n_2 + 2n_3 + 3n_4 + \cdots + 2002n_{2003} \).
6. Let \( a_1 = 1 \), and let \( a_n = \lfloor n^3/a_{n-1} \rfloor \) for \( n > 1 \). Determine the value of \( a_{999} \).
7. Let \( a, b, c \) be the three roots of \( p(x) = x^3 + x^2 - 333x - 1001 \). Find \( a^3 + b^3 + c^3 \).
8. Find the value of \( \frac{1}{\sqrt{x+1}} + \frac{1}{\sqrt{x+2}} + \frac{1}{\sqrt{x+3}} + \cdots \).
9. For how many integers \( n \), for \( 1 \leq n \leq 1000 \), is the number \( \frac{1}{2} \binom{2n}{n} \) even?
10. Suppose \( P(x) \) is a polynomial such that \( P(1) = 1 \) and  
    \[
    \frac{P(2x)}{P(x+1)} = 8 - \frac{56}{x+7}
    \]  
    for all real \( x \) for which both sides are defined. Find \( P(-1) \).