IXth Annual Harvard-MIT Mathematics Tournament
Saturday 25 February 2006

Individual Round: Geometry Test

1. Octagon $ABCDEFGH$ is equiangular. Given that $AB = 1$, $BC = 2$, $CD = 3$, $DE = 4$, and $EF = FG = 2$, compute the perimeter of the octagon.

2. Suppose $ABC$ is a scalene right triangle, and $P$ is the point on hypotenuse $AC$ such that $\angle ABP = 45^\circ$. Given that $AP = 1$ and $CP = 2$, compute the area of $ABC$.

3. Let $A$, $B$, $C$, and $D$ be points on a circle such that $AB = 11$ and $CD = 19$. Point $P$ is on segment $AB$ with $AP = 6$, and $Q$ is on segment $CD$ with $CQ = 7$. The line through $P$ and $Q$ intersects the circle at $X$ and $Y$. If $PQ = 27$, find $XY$.

4. Let $ABC$ be a triangle such that $AB = 2$, $CA = 3$, and $BC = 4$. A semicircle with its diameter on $BC$ is tangent to $AB$ and $AC$. Compute the area of the semicircle.

5. Triangle $ABC$ has side lengths $AB = 2\sqrt{5}$, $BC = 1$, and $CA = 5$. Point $D$ is on side $AC$ such that $CD = 1$, and $F$ is a point such that $BF = 2$ and $CF = 3$. Let $E$ be the intersection of lines $AB$ and $DF$. Find the area of $CDEB$.

6. A circle of radius $t$ is tangent to the hypotenuse, the incircle, and one leg of an isosceles right triangle with inradius $r = 1 + \sin \frac{\pi}{8}$. Find $rt$.

7. Suppose $ABCD$ is an isosceles trapezoid in which $AB \parallel CD$. Two mutually externally tangent circles $\omega_1$ and $\omega_2$ are inscribed in $ABCD$ such that $\omega_1$ is tangent to $AB$, $BC$, and $CD$ while $\omega_2$ is tangent to $AB$, $DA$, and $CD$. Given that $AB = 1$, $CD = 6$, compute the radius of either circle.

8. Triangle $ABC$ has a right angle at $B$. Point $D$ lies on side $BC$ such that $3\angle BAD = \angle BAC$. Given $AC = 2$ and $CD = 1$, compute $BD$.

9. Four spheres, each of radius $r$, lie inside a regular tetrahedron with side length 1 such that each sphere is tangent to three faces of the tetrahedron and to the other three spheres. Find $r$.

10. Triangle $ABC$ has side lengths $AB = 65$, $BC = 33$, and $AC = 56$. Find the radius of the circle tangent to sides $AC$ and $BC$ and to the circumcircle of triangle $ABC$. 