1. Let $S$ be a set of size $n$, and $k$ be a positive integer. For each $1 \leq i \leq kn$, there is a subset $S_i \subset S$ such that $|S_i| = 2$. Furthermore, for each $e \in S$, there are exactly $2k$ values of $i$ such that $e \in S_i$. Show that it is possible to choose one element from $S_i$ for each $1 \leq i \leq kn$ such that every element of $S$ is chosen exactly $k$ times.

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for all real numbers $x, y$,

$$(x - y)(f(x) - f(y)) = f(x - f(y))f(f(x) - y).$$

3. Triangle $ABC$ is inscribed in a circle $\omega$ such that $\angle A = 60^\circ$ and $\angle B = 75^\circ$. Let the bisector of angle $A$ meet $BC$ and $\omega$ at $E$ and $D$, respectively. Let the reflections of $A$ across $D$ and $C$ be $D'$ and $C'$, respectively. If the tangent to $\omega$ at $A$ meets line $BC$ at $P$, and the circumcircle of $APD'$ meets line $AC$ at $F \neq A$, prove that the circumcircle of $C'FE$ is tangent to $BC$ at $E$.

4. A subset $U \subset \mathbb{R}$ is open if for any $x \in U$, there exist real numbers $a, b$ such that $x \in (a, b) \subset U$. Suppose $S \subset \mathbb{R}$ has the property that any open set intersecting $(0, 1)$ also intersects $S$. Let $T$ be a countable collection of open sets containing $S$. Prove that the intersection of all of the sets of $T$ is not a countable subset of $\mathbb{R}$.

(A set $\Gamma$ is countable if there exists a bijective function $f : \Gamma \to \mathbb{Z}$.)

5. (a) Given a finite set $X$ of points in the plane, let $f_X(n)$ be the largest possible area of a polygon with at most $n$ vertices, all of which are points of $X$. Prove that if $m, n$ are integers with $m \geq n > 2$, then $f_X(m) + f_X(n) \geq f_X(m + 1) + f_X(n - 1)$.

(b) Let $P_0$ be a 1-by-2 rectangle (including its interior), and inductively define the polygon $P_i$ to be the result of folding $P_{i-1}$ over some line that cuts $P_{i-1}$ into two connected parts. The diameter of a polygon $P_i$ is the maximum distance between two points of $P_i$. Determine the smallest possible diameter of $P_{2013}$.

(In other words, given a polygon $P_{i-1}$, a fold of $P_{i-1}$ consists of a line $l$ dividing $P_{i-1}$ into two connected parts $A$ and $B$, and the folded polygon $P_i = A \cup B_l$, where $B_l$ is the reflection of $B$ over the line $l$.)