1. **Problem:** Farmer Tim is lost in the densely-forested Cartesian plane. Starting from the origin he walks a sinusoidal path in search of home; that is, after \( t \) minutes he is at position \((t, \sin t)\).

Five minutes after he sets out, Alex enters the forest at the origin and sets out in search of Tim. He walks in such a way that after he has been in the forest for \( m \) minutes, his position is \((m, \cos t)\).

What is the greatest distance between Alex and Farmer Tim while they are walking in these paths?

**Solution:** At arbitrary time \( t \), Farmer Tim is at position \((t, \sin t)\) and Alex is at position \((t - 5, \cos t)\). Hence at time \( t \), the distance, \( d \), between Tim and Alex is \( d = \sqrt{(\sin t - \cos t)^2 + 25} \). To find the maximum value of \( d \), we solve for \( t \) such that \( \frac{dd}{dt} = 0 \).

\[
\frac{dd}{dt} = \frac{(\sin t - \cos t)(\cos t + \sin t)}{\sqrt{(\sin t - \cos t)^2 + 25}}.
\]

Then \( \frac{dd}{dt} = 0 \Rightarrow \sin^2 t - \cos^2 t = 0 \Rightarrow \sin^2 t = \cos^2 t \). Equality happens if \( t \) is any constant multiple of \(\frac{\pi}{4}\).

Notice that to maximize \( d \), we need to maximize \((\sin t - \cos t)^2 \). This is achieved when \( \cos t = -\sin t \). Because we determined earlier that \( t \) is a constant multiple of \(\frac{\pi}{4}\), then with this new condition, we see that \( t \) must be a constant multiple of \(\frac{\pi}{2}\).

Then \((\sin t - \cos t)^2 = 2 \Rightarrow d = \sqrt{25}\).

2. **Problem:** A cube with sides 1m in length is filled with water, and has a tiny hole through which the water drains into a cylinder of radius 1m. If the water level in the cube is falling at a rate of 1 cm/s, at what rate is the water level in the cylinder rising?

**Solution:** The magnitude of the change in volume per unit time of the two solids is the same. The change in volume per unit time of the cube is 1 cm\(^3\)/s. The change in volume per unit time of the cylinder is \(\pi \cdot \frac{dh}{dt} \cdot r^2\), where \(\frac{dh}{dt}\) is the rate at which the water level in the cylinder is rising.

Solving the equation \(\pi \cdot \frac{dh}{dt} \cdot r^2 = 1 \text{ cm} \cdot \text{m}^2/\text{s}\) yields \(\frac{1}{2} \text{ cm/s}\).

3. **Problem:** Find the area of the region bounded by the graphs of \(y = x^2\), \(y = x\), and \(x = 2\).

**Solution:** There are two regions to consider. First, there is the region bounded by \(y = x^2\) and \(y = x\), in the interval \([0, 1]\). In this interval, the values of \(y = x\) are greater than the values of \(y = x^2\), thus the area is calculated by \(\int_0^1 (x - x^2) \, dx\).

Second, there is the region bounded by \(y = x^2\) and \(y = x\) and \(x = 2\), in the interval \([1, 2]\). In this interval, the values of \(y = x^2\) are greater than the values of \(y = x\), thus the area is calculated by \(\int_1^2 (x^2 - x) \, dx\).

Then the total area of the region bounded by the three graphs is \(\int_0^1 (x - x^2) \, dx + \int_1^2 (x^2 - x) \, dx = 1\).

4. **Problem:** Let \(f(x) = 1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \ldots\), for \(-1 \leq x \leq 1\). Find \(\int_0^1 f(x) \, dx\).

**Solution:** Observe that \(f(x)\) is merely an infinite geometric series. Thus \(f(x) = \frac{1}{1 - \frac{x}{2}} = \frac{2}{2-x}\). Then \(\int_0^1 \frac{2}{2-x} = 2 \ln 2\). Then \(\sqrt{\text{e}^{2 \ln 2}} = \sqrt{2^2} = 2\).

5. **Problem:** Evaluate \(\lim_{x \to 1} x^{\frac{\pi}{\sin(1-x)}}\).

**Solution:** Rewrite the expression to evaluate as \(e^{\ln x^{\frac{\pi}{\sin(1-x)}}}\). Then we must evaluate \(\lim_{x \to 1} e^{\ln x^{\frac{\pi}{\sin(1-x)}}}\).

\[\lim_{x \to 1} x^{\frac{\pi}{\sin(1-x)}} = \lim_{x \to 1} \left(\frac{x}{\sin(1-x)} \ln x\right).\]

Because direct calculation of the limit results in indeterminate form \((\frac{1}{0}, 0)\), we can use L’Hopital’s rule to evaluate the limit. By L’Hopital’s rule, \(\lim_{x \to 1} \left(\frac{x}{\sin(1-x)} \ln x\right) = \lim_{x \to 1} \frac{\ln x + 1}{-\cos(1-x)}\). This limit is simply -1.
\[ \lim_{x \to 1} e^{\ln x - \frac{\ln x-1}{x-1}} = e^{-1} = \frac{1}{e} \]

6. **Problem:** Edward, the author of this test, had to escape from prison to work in the grading room today. He stopped to rest at a place 1,875 feet from the prison and was spotted by a guard with a crossbow. The guard fired an arrow with an initial velocity of 100 ft/s. At the same time, Edward started running away with an acceleration of 1 ft/s\(^2\). Assuming that air resistance causes the arrow to decelerate at 1 ft/s\(^2\) and that it does hit Edward, how fast was the arrow moving at the moment of impact (in ft/s)?

**Solution:** We use the formula for distance, \( d = \frac{1}{2}at^2 + vt + d_0 \). Then after \( t \) seconds, Edward is at location \( 1875 + \frac{1}{2}(1)(t^2) \) from the prison. After \( t \) seconds, the arrow is at location \( \frac{1}{2}(-1)(t^2) + 100t \) from the prison. When the arrow hits Edward, both objects are at the same distance away from the tower. Hence \( 1875 + \frac{1}{2}(1)(t^2) = \frac{1}{2}(-1)(t^2) + 100t \). Solving for \( t \) yields \( t^2 - 100t + 1875 = 0 \Rightarrow t = 25 \) or \( t = 75 \). Then it must be \( t = 25 \), because after the arrow hits Edward, he will stop running.

After 25 seconds, the arrow is moving at a velocity of
\[ 100 - 25(1) = \frac{75}{3} \text{ ft/s}. \]

7. **Problem:** A parabola is inscribed in equilateral triangle \( ABC \) of side length 1 in the sense that \( AC \) and \( BC \) are tangent to the parabola at \( A \) and \( B \), respectively. Find the area between \( AB \) and the parabola.

**Solution:** Suppose \( A = (0,0), B = (1,0), \) and \( C = \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right) \). Then the parabola in question goes through \( (0,0) \) and \( (1,0) \) and has tangents with slopes of \( \sqrt{3} \) and \(-\sqrt{3} \), respectively, at these points. Suppose the parabola has equation \( y = ax^2 + bx + c \). Then \( \frac{dy}{dx} = 2ax + b \).

At point \( (0,0) \), \( \frac{dy}{dx} = b \). Also the slope at \( (0,0) \), as we determined earlier, is \( \sqrt{3} \). Hence \( b = \sqrt{3} \). Similarly, at point \( (1,0) \), \( \frac{dy}{dx} = 2a + b \). The slope at \( (1,0) \), as we determined earlier, is \(-\sqrt{3} \). Then \( a = -\sqrt{3} \).

Since the parabola goes through \( (0,0), \) \( c = 0 \). Hence the equation of the parabola is \( y = -\sqrt{3}x^2 + \sqrt{3}x \). The desired area is simply the area under the parabolic curve in the interval \( [0,1] \).

Hence
\[ \int_0^1 \left( -\sqrt{3}x^2 + \sqrt{3}x \right) \, dx = \frac{\sqrt{3}}{6} \]

8. **Problem:** Find the slopes of all lines passing through the origin and tangent to the curve \( y^2 = x^3 + 39x - 35 \).

**Solution:** Any line passing through the origin has equation \( y = mx \), where \( m \) is the slope of the line. If a line is tangent to the given curve, then at the point of tangency, \((x,y)\), \( \frac{dy}{dx} = m \).

First, we calculate \( \frac{dy}{dx} \) of the curve: \( 2y \frac{dy}{dx} = 3x^2 + 39 \Rightarrow \frac{dy}{dx} = \frac{3x^2 + 39}{2y} \). Substituting \( mx \) for \( y \), we get the following system of equations:
\[ m^2 x^2 = x^3 + 39x - 35 \]
\[ m = \frac{3x^2 + 39}{2mx} \]

Solving for \( x \) yields the equation \( x^3 - 39x + 70 = 0 \Rightarrow (x - 2)(x + 7)(x - 5) = 0 \Rightarrow x = 2 \) or \( x = -7 \) or \( x = 5 \). These solutions indicate the \( x \)-coordinate of the points at which the desired lines are tangent to the curve. Solving for the slopes of these lines, we get \( m = \pm \frac{\sqrt{73}}{2} \) for \( x = 2 \), no real solutions for \( x = -7 \), and \( m = \pm \frac{\sqrt{255}}{5} \) for \( x = 5 \). Thus \( m = \pm \frac{\sqrt{73}}{2}, \pm \frac{\sqrt{255}}{5} \).

9. **Problem:** Evaluate \( \sum_{n=1}^{\infty} \frac{1}{n \cdot 2n - 1} \).

**Solution:** Note that if we take the integral of \( f(x) \) in problem 4, we get the function \( F(x) = x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots \). Evaluating this integral in the interval \([0,1]\), we get \( 1 + \frac{1}{2} + \frac{1}{3!} + \ldots \), which is the desired sum.

Hence \( \int_0^1 \frac{2}{2 - x} \, dx = 2 \ln 2 \).
10. **Problem:** Let $S$ be the locus of all points $(x, y)$ in the first quadrant such that \( \frac{x}{t} + \frac{y}{1-t} = 1 \) for some $t$ with $0 < t < 1$. Find the area of $S$.

**Solution:** Solving for $t$ in the given equation, we get \( t^2 + (y - x - 1)t + x = 0 \). Using the quadratic equation, we get \( t = \frac{(x+1-y) \pm \sqrt{(y-x-1)^2-4x}}{2} \). For all valid combinations of $(x, y)$, $t$ is positive and less than 1 (this is easy to see by inspection). All valid combinations of $(x, y)$ are those that make $(y-x-1)^2 - 4x \geq 0$.

Solving for $y$ in the equation $(y-x-1)^2 - 4x = 0$ yields $y^2 - (2x+2)y + (x-1)^2 \geq 0 \Rightarrow y = (x+1) \pm 2\sqrt{x}$. In the original equation, it is given that $\frac{x}{t} + \frac{y}{1-t} = 1$, and $0 < t < 1$. This implies that $x, y < 1$. Then the only possible $y < 1$ that satisfies $(y-x-1)^2 - 4x = 0$ is $y = x + 1 - 2\sqrt{x}$.

Then to satisfy the inequality $(y-x-1)^2 - 4x \geq 0$, we must have $y \leq x + 1 - 2\sqrt{x}$. Recall that this is when $0 < y < 1$. Hence we integrate in the interval $[0, 1]$: $\int_{0}^{1} x + 1 - 2\sqrt{x} = \frac{1}{3}$.