1. \(0.6445\) rounds to \(0.645\) to \(0.65\) to \(0.7\). Otherwise \(0.6444\ldots\) rounds to \(0.644\). So the smallest number is \(\textbf{0.6445}\).

2. Let \(c=\text{price}, p=\text{purity}, d=\text{diameter}, h=\text{depth of gold mine}, k_i=\text{constant.}\) We are given 
\(c = k_1p^2d^3,\) \(p = k_2d^2,\) and \(d = k_3\sqrt[3]{ch}.\) So, \(c = k_1k_2^2k_3^3ch^3 = k_4c^4\). Thus, \(k_4c^4 = h,\) and \(c = k_5h^{\frac{1}{4}}.\) Thus, \(p\) varies as \(h^{\frac{1}{4}}.\)

3. The sum of the numbers from 700 to 799 is \(\frac{700 \cdot 799}{2} - \frac{699 \cdot 700}{2} = 74950.\) The sum of the numbers from 70 to 79 is \(\frac{70 \cdot 79}{2} - \frac{69 \cdot 70}{2} = 745.\) So, all numbers that end from 70 to 79 (excluding those starting with 7, since we counted those from 700 to 799) is \(745 \cdot 9 + 10(100 + 200 + \ldots + 600 + 800 + 900) = 44705.\) The sum of all numbers ending in 7 is \(9(7+17+27+37+47+57+67+77+87+97)+9(100+200+\ldots+600+800+900) = 38187.\) So, the total sum of numbers containing a 7 is \(74950 + 44705 + 38187 = \textbf{157842}.

4. \(\textbf{738,826}.\) This can be arrived at by stepping down, starting with finding how many combinations are there that begin with a letter other than V or W, and so forth. The answer is \(\frac{8}{22} + \frac{4}{22} + 4 \cdot 6! + 4 \cdot 4! + 3! + 2! + 2! = \textbf{738826}.

5. \(0 = \cos (\alpha + \beta) + \sin (\alpha - \beta) = \cos \alpha \cdot \cos \beta - \sin \alpha \cdot \sin \beta + \sin \alpha \cdot \cos \beta - \sin \beta \cos \alpha = (\cos \alpha + \sin \alpha) \cdot (\cos \beta - \sin \beta).\) So \(\cos \alpha + \sin \alpha = 0\) or \(\cos \beta - \sin \beta = 0.\) Then \(\tan \alpha = -1\) or \(\tan \beta = 1.\) Since \(\tan \beta\) is given as \(\frac{1}{\sin \beta},\) \(\tan \alpha = -1.\)

6. Since \(\alpha^3 - 2\alpha - 1 = 0,\) then \(\alpha^{10} = \alpha^8 + \alpha^7.\) So, we can reduce our expression to \(3\alpha^8 - 3\alpha^6 - 3\alpha^5 + 4\alpha^4 + 2\alpha^3 - 4\alpha^2 - 6\alpha - 17.\) Also, \(3\alpha^8 - 3\alpha^6 - 3\alpha^5 = 0,\) so our expression reduces to \(4\alpha^4 + 2\alpha^3 - 4\alpha^2 - 6\alpha - 17.\) Also, \(4\alpha^4 - 4\alpha^2 - 2 \alpha^6,\) so our expression reduces to \(2\alpha^3 - 2\alpha - 17.\) Now, \(2\alpha^3 - 2\alpha - 2 = 0,\) so our expression reduces to \(-15,\) which is our answer.

7. Another 4-digit number that satisfies this property is \(9801,\) since \(9801=9\times1089.\)

8. If she has a silver dollar, then she would have too many other coins, as 0 half dollars, 2 quarters, 3 dimes, etc. would be greater than the total. She has no silver dollars, and at least one of every other denomination. Continuing, it turns out the only feasible solution is 0 silver dollars, 1 half dollar, 2 quarters, 3 dimes, 4 nickels, 8 pennies, for a total of \(18\) coins.

9. It suffices to consider \(x \geq 1,\) since \(4(-x)^4 + 1 = 4(x)^4 + 1,\) and \(4(0) + 1 = 1\) is not prime. So, \(4x^4 + 1 = (4x^4 + 4x^2 + 1) - 4x^2 = (2x^2 + 1)^2 - (2x)^2 = (2x^2 + 1 - 2x)(2x^2 + 1 + 2x).\) For integers \(x,\) both \(2x^2 - 2x + 1\) and \(2x^2 + 2x + 1\) are integers, so this factors \(4x^4 + 1\) unless \(2x^2 - 2x + 1 = \pm 1\) or \(2x^2 + 2x + 1 = \pm 1.\) Since \(x > 0,\) then \(2x^2 + 2x + 1 > 1,\) so we must have \(2x^2 - 2x + 1 = \pm 1.\) \(2x^2 - 2x + 1 = -1\) is absurd \((4x^4 + 1, 2x^2 + 2x + 1 > 0,\) so \(2x^2 - 2x + 1 = \frac{4x^4 + 1}{2x^2 + 2x + 1} > 0,\) so we solve \(2x^2 - 2x + 1 = 1,\) or \(2x^2 - 2x = 0,\) so \(x(x - 1) = 0,\) and \(x = 0\) or \(x = 1.\) We have already rejected \(x = 0,\) so the only case left is \(x = 1,\) or \(4(1)^4 + 1 = 5.\)

10. The second hand crosses the minute hand \(59\) times an hour. The second hand crosses the hour hand \(60\) times an hour, except for \(2\) of the hours, due to the movement of the hour hand. The minute hand and the hour hand cross \(22\) times total, because the hour hand
completes 2 rotations in a day, and the minute hand completes 24. The second, hour, and minute hand all coincide only at noon and midnight, but we’ve counted each of these 12:00’s 3 times instead of once. Therefore, the answer is 59·60+60·24−2+22−2·2, giving us 2872 crossings.

11. \( f(x) \) is either 0 or something of the form \( \pm a^m \), where \( m \geq 0 \).

12. \( A \)'s position is \( (a - V_a t, 0) \) and \( P \)'s position is \( (0, b - V_b t) \). So, at time \( t \), the distance between them is \( \sqrt{(a - V_a t)^2 + (b - V_b t)^2} \). Notice that this distance is the same as the distance between the point \( (a, b) \) and the line \( (V_a t, V_b t) \), which is the same as the line \( V_b x - V_a y = 0 \). The distance from a line \( Ax + By + C = 0 \) and \( (x_0, y_0) \) is \( \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}} \), so the answer is \( \frac{\sqrt{V_a^2 + V_b^2}}{V_a^2 + V_b^2} \).

13. Given any 4 vertices, there is exactly one intersection of all the diagonals connecting them. So, the answer is \( \binom{n}{4} \).

14. \( x_0 \) if \( n \equiv 0 \pmod{4} \), \( \frac{1 + 2n}{1 - 2n} \) if \( n \equiv 1 \pmod{4} \), \( -\frac{1}{2n} \) if \( n \equiv 2 \pmod{4} \), \( \frac{2n + 1}{2n + 1} \) if \( n \equiv 3 \pmod{4} \).

15. Consider a regular \( n \)-gon with radius \( r \). Let \( x \) be the side length of the \( n \)-gon. So, since the central angle is \( \frac{2\pi}{n} \) (see diagram below), use the Law of Cosines to find that \( x^2 = r^2 + r^2 - 2r \cdot r \cos \frac{2\pi}{n} \), so \( x^2 = 2r^2(1 - \cos \frac{2\pi}{n}) \). Thus, \( x = r \sqrt{\frac{2r^2}{1 - \cos \frac{2\pi}{n}}} \). So, the total perimeter of the \( n \)-gon is \( n\cdot x = nr \cdot \sqrt{\frac{2}{1 - \cos \frac{2\pi}{n}}} \). Now, if we take \( \lim_{n \to \infty} \) of the perimeter, the result will be \( 2\pi r \), since the \( n \)-gon approaches a circle, so \( \lim_{n \to \infty} nk \sqrt{\frac{2}{1 - \cos \frac{2\pi}{n}}} = 2\pi r \), and so \( \lim_{n \to \infty} nk \sqrt{1 - \cos \frac{2\pi}{n}} = \pi r \sqrt{2} \).

![Diagram](image-url)