Algebra Test Solutions

Harvard-MIT Math Tournament
March 3, 2001

1. Find $x - y$, given that $x^4 = y^4 + 24$, $x^2 + y^2 = 6$, and $x + y = 3$.
Solution: $\frac{24}{63} = \frac{x^4 - y^4}{(x^2 + y^2)(x + y)} = \frac{(x^2 + y^2)(x + y)}{(x^2 + y^2)(x + y)} = x - y = \frac{4}{3}$

2. Find $(x + 1)(x^2 + 1)(x^4 + 1)(x^8 + 1) \cdots$, where $|x| < 1$.
Solution: Let $S = (x + 1)(x^2 + 1)(x^8 + 1) \cdots = 1 + x + x^2 + x^3 + \cdots$. Since $xS = x + x^2 + x^3 + x^4 + \cdots$, we have $(1 - x)S = 1$, so $S = \frac{1}{1-x}$.

3. How many times does 24 divide into 100! (factorial)?
Solution: We first determine the number of times 2 and 3 divide into 100! = 1·2·3·⋯·100. Let $\langle N \rangle_n$ be the number of times $n$ divides into $N$ (i.e. we want to find $\langle 100! \rangle_{24}$). Since 2 only divides into even integers, $\langle 100! \rangle_2 = (2 \cdot 4 \cdot 6 \cdots 100)$. Factoring out 2 once from each of these multiples, we get that $\langle 100! \rangle_2 = (2^{50} \cdot 1 \cdot 2 \cdot 3 \cdots 50)_2$. Repeating this process, we find that $\langle 100! \rangle_2 = (20^{50} + 25 + 12 + 6 + 3 + 1 \cdot 1)_2 = 97$. Similarly, $\langle 100! \rangle_3 = (3^{33} + 11 + 3 + 1)_3 = 48$. Now $24 = 2^3 \cdot 3$, so for each factor of 24 in 100! there needs to be three multiples of 2 and one multiple of 3 in 100!. Thus $\langle 100! \rangle_{24} = [(\langle 100! \rangle_2 / 3) + \langle 100! \rangle_3] = 32$, where $[N]$ is the greatest integer less than or equal to $N$.

4. Given that 7, 999, 999, 999 has at most two prime factors, find its largest prime factor.

5. Find the 6-digit number beginning and ending in the digit 2 that is the product of three consecutive even integers.
Solution: Because the last digit of the product is 2, none of the three consecutive even integers end in 0. Thus they must end in 2, 4, 6 or 4, 6, 8, so they must end in 4, 6, 8 since 2-4-6 does not end in 2. Call the middle integer $n$. Then the product is $(n - 2)n(n + 2) = n^3 - 4n$, so $n > \sqrt[3]{2000000} = \sqrt[3]{200 \cdot 10^3} \approx 60$, but clearly $n < \sqrt[3]{300000} = \sqrt[3]{300 \cdot 10^3} < 70$. Thus $n = 66$, and the product is 66³ - 4 · 66 = 287232.

6. What is the last digit of 1¹ + 2² + 3³ + ⋯ + 100¹⁰⁰?
Solution: Let $L(d, n)$ be the last digit of a number ending in $d$ to the $n$th power. For $n \geq 1$, we know that $L(0, n) = 0$, $L(1, n) = 1$, $L(5, n) = 5$, $L(6, n) = 6$. All numbers ending in odd digits in this series are raised to odd powers; for odd $n$, $L(3, n) = 3$ or 7, $L(7, n) = 3$ or 7, $L(9, n) = 9$. All numbers ending in even digits are raised to even powers; for even $n$, $L(2, n) = 4$ or 6, $L(4, n) = L(6, n) = 6$, $L(8, n) = 6$ or 4. Further, for each
last digit that has two possible values, the possible values will be present equally as often. 

Now define $S(d)$ such that $S(0) = 0$ and for $1 \leq d \leq 9$, $S(d) = L(d,d) + L(d,d + 10) + L(d,d + 20) + L(d,d + 30) + \cdots + L(d,d + 90)$, so that the sum we want to calculate becomes $S(0) + S(1) + S(2) + \cdots + S(9)$. But by the above calculations all $S(d)$ are divisible by 10, so their sum is divisible by 10, which means its last digit is 0.

7. A polynomial $P$ has four roots, $\frac{1}{4}, \frac{1}{2}, 2, 4$. The product of the roots is 1, and $P(1) = 1$. Find $P(0)$.

Solution: A polynomial $Q$ with $n$ roots, $x_1, \ldots, x_n$, and $Q(x_0) = 1$ is given by $Q(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}$, so $P(0) = \frac{1}{\frac{1}{4} \cdot \frac{1}{2} \cdot (-1) \cdot (-3)} = \frac{8}{9}$

8. How many integers between 1 and 2000 inclusive share no common factors with 2001?

Solution: Two integers are said to be relatively prime if they share no common factors, that is if there is no integer greater than 1 that divides evenly into both of them. Note that 1 is relatively prime to all integers. Let $\varphi(n)$ be the number of integers less than $n$ that are relatively prime to $n$. Since $\varphi(mn) = \varphi(m)\varphi(n)$ for $m$ and $n$ relatively prime, we have $\varphi(2001) = \varphi(3 \cdot 23 \cdot 29) = (3 - 1)(23 - 1)(29 - 1) = 1232$

9. Find the number of positive integer solutions to $n^x + n^y = n^z$ with $n^z < 2001$.

Solution: If $n = 1$, the relation cannot hold, so assume otherwise. If $x > y$, the left hand side factors as $n^y(n^{x-y} + 1)$ so $n^{x-y} + 1$ is a power of $n$. But it leaves a remainder of 1 when divided by $n$ and is greater than 1, a contradiction. We reach a similar contradiction if $y > x$. So $y = x$ and $2n^x = n^z$, so 2 is a power of $n$ and $n = 2$. So all solutions are of the form $2^x + 2^x = 2^{x+1}$, which holds for all $x$. $2^{x+1} < 2001$ implies $x < 11$, so there are 10 solutions.

10. Find the real solutions of $(2x + 1)(3x + 1)(5x + 1)(15x + 1) = 10$.

Solution: $(2x + 1)(3x + 1)(5x + 1)(15x + 1) = [(2x + 1)(30x + 1)][(3x + 1)(5x + 1)] = (60x^2 + 32x + 1)(15x^2 + 8x + 1) = (4y + 1)(y + 1) = 10$, where $y = 15x^2 + 8x$. The quadratic equation in $y$ yields $y = 1$ and $y = -\frac{9}{4}$. For $y = 1$, we have $15x^2 + 8x - 1 = 0$, so $x = \frac{-4 \pm \sqrt{51}}{15}$. For $y = -\frac{9}{4}$, we have $15x^2 + 8x + \frac{9}{4}$, which yields only complex solutions for $x$. Thus the real solutions are $\frac{-4 \pm \sqrt{51}}{15}$. 