1. Two circles have centers that are \( d \) units apart, and each has diameter \( \sqrt{d} \). For any \( d \), let \( A(d) \) be the area of the smallest circle that contains both of these circles. Find \( \lim_{d \to \infty} \frac{A(d)}{d^2} \).

**Solution:** This equals \( \lim_{d \to \infty} \frac{\pi \left( \frac{d + \sqrt{d}}{2} \right)^2}{d^2} = \frac{\pi}{4} \).

2. Find \( \lim_{h \to 0} \frac{x^2 - (x + h)^2}{h} \).

**Solution:** This equals \( \lim_{h \to 0} \frac{x^2 - x^2 - h^2 - 2hx}{h} = \lim_{h \to 0} -2x = -2x \). Alternate Solution: This is the definition of the derivative of \( -x^2 \) with respect to \( x \), which is \(-2x\).

3. We are given the values of the differentiable real functions \( f, g, h \), as well as the derivatives of their pairwise products, at \( x = 0 \):

\[
f(0) = 1; \quad g(0) = 2; \quad h(0) = 3; \quad (gh)'(0) = 4; \quad (hf)'(0) = 5; \quad (fg)'(0) = 6.
\]

Find the value of \( (fg'h)'(0) \).

**Solution:** \([16] \) By the product rule, \( (fg'h)' = f'gh + fg'h + fgh' = ((fg)'h + (gh)'f + (hf)'g)/2 \). Evaluated at 0, this gives 16.

4. Find the area of the region in the first quadrant \( x > 0, y > 0 \) bounded above the graph of \( y = \arcsin(x) \) and below the graph of the \( y = \arccos(x) \).

**Solution:** We can integrate over \( y \) rather than \( x \). In particular, the solution is \( \int_0^{\pi/4} \sin y \, dy + \int_{\pi/4}^{\pi/2} \cos y \, dy = \left(1 - \frac{1}{\sqrt{2}}\right)2 = 2 - \sqrt{2} \).

5. What is the minimum vertical distance between the graphs of \( 2 + \sin(x) \) and \( \cos(x) \)?

**Solution:** The derivative of \( 2 + \sin(x) - \cos(x) \) is \( \cos x + \sin x \), which in the interval \( 0 \leq x < 2\pi \) is zero at \( x = \frac{3\pi}{4}, \frac{5\pi}{4} \). At \( \frac{7\pi}{4} \), when \( \sin(x) \) is negative and \( \cos(x) \) is positive, the distance reaches its minimal value of \( \frac{2}{\sqrt{2}} \).

6. Determine the positive value of \( a \) such that the parabola \( y = x^2 + 1 \) bisects the area of the rectangle with vertices \((0, 0), (a, 0), (0, a^2 + 1), \) and \((a, a^2 + 1)\).

**Solution:** \( \sqrt{3} \) The area of the rectangle is \( a^3 + a \). The portion under the parabola has area \( \int_0^a x^2 + 1 \, dx = a^3/3 + a \). Thus we wish to solve the equation \( a^3 + a = 2(a^3/3 + a) \); dividing by \( a \) and rearranging gives \( a^2/3 = 1 \), so \( a = \sqrt{3} \).
7. Denote by $\langle x \rangle$ the fractional part of the real number $x$ (for instance, $\langle 3.2 \rangle = 0.2$). A positive integer $N$ is selected randomly from the set $\{1, 2, 3, \ldots, M\}$, with each integer having the same probability of being picked, and $\langle \frac{N}{M} \rangle$ is calculated. This procedure is repeated $M$ times and the average value $A(M)$ is obtained. What is $\lim_{M \to \infty} A(M)$?

**Solution:** This method of picking $N$ is equivalent to uniformly randomly selecting a positive integer. Call this the average value of $\langle \frac{N}{M} \rangle$ for $N$ a positive integer. In lowest terms, $\frac{87}{303} = \frac{29}{101}$, so the answer is the same as the average value of $\frac{0}{101}, \frac{1}{101}, \ldots, \frac{100}{101}$, which is $\frac{50}{101}$.

8. Evaluate $\int_0^{(\sqrt{2}-1)/2} \frac{dx}{(2x+1)^{\sqrt{2}+x}}$.

**Solution:** Let $u = \sqrt{x^2 + x}$. Then $du = \frac{2x+1}{2\sqrt{x^2+x}} \, dx$. So the integral becomes $2 \int \frac{du}{(4u^2+4u+1)}$, or $2 \int \frac{du}{4u^2+4}$. This is $\tan^{-1}(2u)$, yielding a final answer of $\tan^{-1}(2) + C$ for the indefinite integral. The definite integral becomes $\tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4}$.

9. Suppose $f$ is a differentiable real function such that $f(x) + f'(x) \leq 1$ for all $x$, and $f(0) = 0$. What is the largest possible value of $f(1)$? (Hint: consider the function $e^x f(x)$.)

**Solution:** Let $g(x) = e^x f(x)$; then $g'(x) = e^x (f(x) + f'(x)) \leq e^x$. Integrating from 0 to 1, we have $g(1) - g(0) = \int_0^1 g'(x) \, dx \leq \int_0^1 e^x \, dx = e - 1$. But $g(1) - g(0) = e \cdot f(1)$, so we get $f(1) \leq (e - 1)/e$. This maximum is attained if we actually have $g'(x) = e^x$ everywhere; this entails the requirement $f(x) + f'(x) = 1$, which is met by $f(x) = 1 - e^{-x}$.

10. A continuous real function $f$ satisfies the identity $f(2x) = 3f(x)$ for all $x$. If $\int_0^1 f(x) \, dx = 1$, what is $\int_0^1 f^2(x) \, dx$?

**Solution:** Let $S = \int_0^1 f(x) \, dx$. By setting $u = 2x$, we see that $\int_0^1 f(x) \, dx = \int_{1/2}^1 f(2u)/3 \, du = \int_0^1 f(u)/6 \, du = S/6$. Similarly, $\int_{1/4}^{1/2} f(x) \, dx = S/36$, and in general $\int_{1/2^n}^{1/2^{n-1}} f(x) \, dx = S/6^n$. Adding finitely many of these, we have $\int_0^1 f(x) \, dx = S/6 + S/36 + \cdots + S/6^n = S \cdot (1 - 1/6^n)/6$. Taking the limit as $n \to \infty$, we have $\int_0^1 f(x) \, dx = S/5$. Thus $S = 5$, the answer.