10th Annual Harvard-MIT Mathematics Tournament
Saturday 24 February 2007

Individual Round: Algebra Test

1. [3] Compute

\[ \frac{2007! + 2004!}{2006! + 2005!}. \]

(Note that \( \lfloor x \rfloor \) denotes the greatest integer less than or equal to \( x \).)

**Answer:** 2006. We have

\[ \frac{2007! + 2004!}{2006! + 2005!} = \frac{(2007 \cdot 2006 + \frac{1}{2005}) \cdot 2005!}{(2006 + 1) \cdot 2005!} = \frac{2007 \cdot 2006 + \frac{1}{2005}}{2007} = \frac{2006 + \frac{1}{2005 \cdot 2007}}. \]

2. [3] Two reals \( x \) and \( y \) are such that \( x - y = 4 \) and \( x^3 - y^3 = 28 \). Compute \( xy \).

**Answer:** -3. We have 28 = \( x^3 - y^3 = (x - y)(x^2 + xy + y^2) = (x - y)((x - y)^2 + 3xy) = 4 \cdot (16 + 3xy) \), from which \( xy = -3 \).

3. [4] Three real numbers \( x, y, \) and \( z \) are such that \( (x + 4)/2 = (y + 9)/(z - 3) = (x + 5)/(z - 5) \). Determine the value of \( x/y \).

**Answer:** 1/2. Because the first and third fractions are equal, adding their numerators and denominators produces another fraction equal to the others: \( ((x + 4) + (x + 5))/((2 + (z - 5)) = (2x + 9)/(z - 3) \). Then \( y + 9 = 2x + 9 \), etc.


\[ \frac{2^3 - 1}{3^3 + 1} \cdot \frac{3^3 - 1}{4^3 + 1} \cdot \frac{4^3 - 1}{5^3 + 1} \cdot \frac{5^3 - 1}{6^3 + 1}. \]

**Answer:** \[ \frac{43}{63} \]

Use the factorizations \( n^3 - 1 = (n - 1)(n^2 + n + 1) \) and \( n^3 + 1 = (n + 1)(n^2 - n + 1) \) to write

\[ \frac{1 \cdot 7}{3 \cdot 3} \cdot \frac{2 \cdot 13}{4 \cdot 7} \cdot \frac{3 \cdot 21}{5 \cdot 13} \cdot \frac{4 \cdot 31}{6 \cdot 21} \cdot \frac{5 \cdot 43}{7 \cdot 31} = \frac{1 \cdot 2 \cdot 43}{3 \cdot 6 \cdot 7} = \frac{43}{63}. \]

5. [5] A convex quadrilateral is determined by the points of intersection of the curves \( x^4 + y^4 = 100 \) and \( xy = 4 \); determine its area.

**Answer:** \[ 4\sqrt{17} \]. By symmetry, the quadrilateral is a rectangle having \( x = y \) and \( x = -y \) as axes of symmetry. Let \( \{a, b\} \) with \( a > b > 0 \) be one of the vertices. Then the desired area is

\[ (\sqrt{2}(a - b)) \cdot (\sqrt{2}(a + b)) = 2(a^2 - b^2) = 2\sqrt{a^4 - 2a^2b^2 + b^4} = 2\sqrt{100 - 2 \cdot 4^2} = 4\sqrt{17}. \]

6. [5] Consider the polynomial \( P(x) = x^3 + x^2 - x + 2 \). Determine all real numbers \( r \) for which there exists a complex number \( z \) not in the reals such that \( P(z) = r \).

**Answer:** \( r > 3, r < \frac{49}{27} \). Because such roots to polynomial equations come in conjugate pairs, we seek the values \( r \) such that \( P(x) = r \) has just one real root \( x \). Considering the shape of a cubic, we are interested in the boundary values \( r \) such that \( P(x) = r \) has a repeated zero. Thus, we write

\[ P(x) - r = x^3 + x^2 - x + (2 - r) = (x - p)^2(x - q) = x^3 - (2p + q)x^2 + p(p + 2q)x - p^2q. \]

Then \( q = -2p -1 \) and \( 1 = p(p + 2q) = p(-3p - 2) \) so that \( p = 1/3 \) or \( p = -1 \). It follows that the graph of \( P(x) \) is horizontal at \( x = 1/3 \) (a maximum) and \( x = -1 \) (a minimum), so the desired values \( r \) are \( r > P(-1) = 3 \) and \( r < P(1/3) = 1/27 + 1/9 - 1/3 + 2 = 49/27. \)
7. [5] An infinite sequence of positive real numbers is defined by \( a_0 = 1 \) and \( a_{n+2} = 6a_n - a_{n+1} \) for \( n = 0, 1, 2, \ldots \). Find the possible value(s) of \( a_{2007} \).

**Answer:** \( 2^{2007} \). The characteristic equation of the linear homogeneous equation is \( m^2 + m - 6 = (m + 3)(m - 2) = 0 \) with solutions \( m = -3 \) and \( m = 2 \). Hence the general solution is given by \( a_n = A(2)^n + B(-3)^n \) where \( A \) and \( B \) are constants to be determined. Then we have \( a_n > 0 \) for \( n \geq 0 \), so necessarily \( B = 0 \), and \( a_0 = 1 \Rightarrow A = 1 \). Therefore, the unique solution to the recurrence is \( a_n = 2^n \) for all \( n \).

8. [6] Let \( A := \mathbb{Q} \setminus \{0, 1\} \) denote the set of all rationals other than 0 and 1. A function \( f : A \to \mathbb{R} \) has the property that for all \( x \in A \),

\[
f(x) + f \left( 1 - \frac{1}{x} \right) = \log |x|.
\]

Compute the value of \( f(2007) \).

**Answer:** \( \log(2007/2006) \). Let \( g : A \to A \) be defined by \( g(x) := 1 - 1/x \); the key property is that

\[
g(g(x)) = 1 - \frac{1}{1 - \frac{1}{x}} = x.
\]

The given equation rewrites as \( f(x) + f(g(x)) = \log |x| \). Substituting \( x = g(y) \) and \( x = g(z) \) gives the further equations \( f(g(y)) + f(g(g(y))) = \log |x| \) and \( f(g(z)) + f(g(g(z))) = \log |x| \). Setting \( y \) and \( z \) to \( x \) and solving the system of three equations for \( f(x) \) gives

\[
f(x) = \frac{1}{2} \left( \log |x| - \log |g(x)| - \log |g(g(x))| \right) .
\]

For \( x = 2007 \), we have \( g(x) = \frac{2006}{2007} \) and \( g(g(x)) = \frac{-1}{2006} \), so that

\[
f(2007) = \frac{\log |2007| - \log \left| \frac{2006}{2007} \right| + \log \left| \frac{-1}{2006} \right|}{2} = \log \left( \frac{2007}{2006} \right) .
\]

9. [7] The complex numbers \( \alpha_1, \alpha_2, \alpha_3, \) and \( \alpha_4 \) are the four distinct roots of the equation \( x^4 + 2x^3 + 2 = 0 \). Determine the unordered set

\[
\{ \alpha_1\alpha_2 + \alpha_3\alpha_4, \alpha_1\alpha_3 + \alpha_2\alpha_4, \alpha_1\alpha_4 + \alpha_2\alpha_3 \}.
\]

**Answer:** \( \{ 1 \pm \sqrt{5}, -2 \} \). Employing the elementary symmetric polynomials \( s_1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = -2, s_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4 = 0, s_3 = \alpha_1\alpha_2\alpha_3 + \alpha_2\alpha_3\alpha_4 + \alpha_3\alpha_4\alpha_1 + \alpha_4\alpha_1\alpha_2 = 0, \) and \( s_4 = \alpha_1\alpha_2\alpha_3\alpha_4 = 2 \) we consider the polynomial

\[
P(x) = (x - (\alpha_1\alpha_2 + \alpha_3\alpha_4))(x - (\alpha_1\alpha_3 + \alpha_2\alpha_4))(x - (\alpha_1\alpha_4 + \alpha_2\alpha_3))
\]

Because \( P \) is symmetric with respect to \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \), we can express the coefficients of its expanded form in terms of the elementary symmetric polynomials. We compute

\[
P(x) &= x^3 - 8x - 8
\]

\[
= (x + 2)(x^2 - 2x - 4)
\]

The roots of \( P(x) \) are \(-2\) and \( 1 \pm \sqrt{5} \), so the answer is \( \{1 \pm \sqrt{5}, -2\} \).

**Remarks.** It is easy to find the coefficients of \( x^2 \) and \( x \) by expansion, and the constant term can be computed without the complete expansion and decomposition of \( (\alpha_1\alpha_2 + \alpha_3\alpha_4)(\alpha_1\alpha_3 + \alpha_2\alpha_4)(\alpha_1\alpha_4 + \alpha_2\alpha_3) \) by noting that the only nonzero 6th degree expressions in \( s_1, s_2, s_3, \) and \( s_4 \) are \( s_1^6 \) and \( s_4s_1^5 \). The general polynomial \( P \) constructed here is called the cubic resolvent and arises in Galois theory.
10. The polynomial \( f(x) = x^{2007} + 17x^{2006} + 1 \) has distinct zeroes \( r_1, \ldots, r_{2007} \). A polynomial \( P \) of degree 2007 has the property that \( P \left( r_j + \frac{1}{r_j} \right) = 0 \) for \( j = 1, \ldots, 2007 \). Determine the value of \( P(1)/P(-1) \).

**Answer:** For some constant \( k \), we have

\[
P(z) = k \prod_{j=1}^{2007} \left( z - \left( r_j + \frac{1}{r_j} \right) \right).
\]

Now writing \( \omega^3 = 1 \) with \( \omega \neq 1 \), we have \( \omega^2 + \omega = -1 \). Then

\[
P(1)/P(-1) = \frac{f(-\omega)f(-\omega^2)}{f(\omega)f(\omega^2)} = \frac{\left( -\omega^{2007} + 17\omega^{2006} + 1 \right) \left( -(\omega^2)^{2007} + 17(\omega^2)^{2006} + 1 \right)}{\left( \omega^{2007} + 17\omega^{2006} + 1 \right) \left( (\omega^2)^{2007} + 17(\omega^2)^{2006} + 1 \right)} = \frac{(17\omega^3)(17\omega)}{289} = \frac{289}{289} = 1.\]