10th Annual Harvard-MIT Mathematics Tournament
Saturday 24 February 2007

Individual Round: Calculus Test

1. [3] Compute:

\[ \lim_{x \to 0} \frac{x^2}{1 - \cos(x)} \]

**Answer:** Since \( \sin^2(x) = 1 - \cos^2(x) \), we multiply the numerator and denominator by \( 1 + \cos(x) \) and use the fact that \( x / \sin(x) \to 1 \), obtaining

\[ \lim_{x \to 0} \frac{x^2}{1 - \cos(x)} = \lim_{x \to 0} \frac{x^2(1 + \cos(x))}{1 - \cos^2(x)} = \lim_{x \to 0} \left( \frac{x}{\sin(x)} \right)^2 \cdot 2 = 2 \]

**Remarks.** Another solution, using \( \text{L'Hôpital's rule} \), is possible: \( \lim_{x \to 0} \frac{x^2}{1 - \cos(x)} = \lim_{x \to 0} \frac{2x}{2 \sin(x)} = 2 \).

2. [3] Determine the real number \( a \) having the property that \( f(a) = a \) is a relative minimum of \( f(x) = x^4 - x^3 - x^2 + ax + 1 \).

**Answer:** Being a relative minimum, we have \( 0 = f'(a) = 4a^3 - 3a^2 - 2a + a = a(4a^2 + 1)(a - 1) \). Then \( a = 0, 1, -1/4 \) are the only possibilities. However, it is easily seen that \( a = 1 \) is the only value satisfying \( f(a) = a \).

3. [4] Let \( a \) be a positive real number. Find the value of \( a \) such that the definite integral

\[ \int_a a^2 \frac{dx}{x + \sqrt{x}} \]

achieves its smallest possible value.

**Answer:** \( 3 - 2\sqrt{2} \). Let \( F(a) \) denote the given definite integral. Then

\[ F'(a) = \frac{d}{da} \int_a a^2 \frac{dx}{x + \sqrt{x}} = 2a \cdot \frac{1}{a^2 + \sqrt{a}^2} - \frac{1}{a + \sqrt{a}}. \]

Setting \( F'(a) = 0 \), we find that \( 2a + 2\sqrt{a} = a + 1 \) or \( (\sqrt{a} + 1)^2 = 2 \). We find \( \sqrt{a} = \pm \sqrt{2} - 1 \), and because \( \sqrt{a} > 0 \), \( a = (\sqrt{2} - 1)^2 = 3 - 2\sqrt{2} \).

4. [4] Find the real number \( \alpha \) such that the curve \( f(x) = e^x \) is tangent to the curve \( g(x) = ax^2 \).

**Answer:** \( e^{\alpha^2/4} \). Suppose tangency occurs at \( x = x_0 \). Then \( e^x_0 = \alpha x_0^2 \) and \( f'(x_0) = 2\alpha x_0 \). On the other hand, \( f'(x) = f(x) \), so \( \alpha x_0^2 = 2\alpha x_0 \). Clearly, \( \alpha = 0 \) and \( x_0 = 0 \) are impossible, so it must be that \( x_0 = 2 \). Then \( \alpha = e^{x_0^2/2} = e^{\alpha^2/4} \).

5. [5] The function \( f : \mathbb{R} \to \mathbb{R} \) satisfies \( f(x^2)f''(x) = f'(x)f'(x^2) \) for all real \( x \). Given that \( f(1) = 1 \) and \( f''(1) = 8 \), determine \( f'(1) + f''(1) \).

**Answer:** \( 6 \). Let \( f'(1) = a \) and \( f''(1) = b \). Then setting \( x = 1 \) in the given equation, \( b = a^2 \). Differentiating the given yields

\[ 2xf'(x^2)f''(x) + f(x^2)f'''(x) = f''(x)f'(x^2) + 2xf'(x)f''(x^2). \]

Plugging \( x = 1 \) into this equation gives \( 2ab + 8 = ab + 2ab \), or \( ab = 8 \). Then because \( a \) and \( b \) are real, we obtain the solution \( (a, b) = (2, 4) \).

**Remarks.** A priori, the function needn't exist, but one possibility is \( f(x) = e^{2x-2} \).
6. [5] The elliptic curve \( y^2 = x^3 + 1 \) is tangent to a circle centered at \((4, 0)\) at the point \((x_0, y_0)\). Determine the sum of all possible values of \(x_0\).

**Answer:** \(\frac{1}{3}\) Note that \(y^2 \geq 0\), so \(x^3 \geq -1\) and \(x \geq -1\). Let the circle be defined by \((x-4)^2 + y^2 = c\) for some \(c \geq 0\). Now differentiate the equations with respect to \(x\), obtaining \(2y \frac{dy}{dx} = 3x^2\) from the given and \(2y \frac{dy}{dx} = -2x + 8\) from the circle. For tangency, the two expressions \( \frac{dy}{dx} \) must be equal if they are well-defined, and this is almost always the case. Thus, \(-2x_0 + 8 = 3x_0^2\) so \(x_0 = -2\) or \(x_0 = 4/3\), but only the latter corresponds to a point on \(y^2 = x^3 + 1\). Otherwise, \(y_0 = 0\), and this gives the trivial solution \(x_0 = -1\).

7. [5] Compute

\[
\sum_{n=1}^{\infty} \frac{1}{n \cdot (n+1) \cdot (n+1)!} = \frac{1}{n} - \frac{1}{n+1} \quad \text{for some} \quad c
\]

**Answer:** \(3 - e\) We write

\[
\sum_{n=1}^{\infty} \frac{1}{n \cdot (n+1) \cdot (n+1)!} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{n+1} \cdot (n+1)! = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n \cdot (n+1)!} \quad \text{for some} \quad c
\]

\[
\frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n \cdot (n+1)!} = \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \cdots = 3 - \frac{1}{0! + 1! + \frac{1}{2!} + \cdots} = 3 - e.
\]

Alternatively, but with considerably less motivation, we can induce telescoping by adding and subtracting \(e - 2 = 1/2! + 1/3! + \cdots\), obtaining

\[
2 - e + \sum_{n=1}^{\infty} \frac{n(n+1) + 1}{n \cdot (n+1) \cdot (n+1)!} = 2 - e + \sum_{n=1}^{\infty} \frac{(n+1)^2 - n}{n \cdot (n+1) \cdot (n+1)!} = 2 - e + \sum_{n=1}^{\infty} \frac{1}{n \cdot n!} - \frac{1}{(n+1) \cdot (n+1)!} = 3 - e.
\]

8. [6] Suppose that \(\omega\) is a primitive 2007\(^{th}\) root of unity. Find \((2^{2007} - 1) \sum_{j=1}^{2006} \frac{1}{2^j}\).

For this problem only, you may express your answer in the form \(m \cdot n^k + p\), where \(m, n, k, \) and \(p\) are positive integers. Note that a number \(z\) is a **primitive \(n^{th}\) root of unity** if \(z^n = 1\) and \(n\) is the smallest number amongst \(k = 1, 2, \ldots, n\) such that \(z^k = 1\).

**Answer:** \(2005 \cdot 2^{2006} + 1\) Note that

\[
\frac{1}{z - \omega} + \cdots + \frac{1}{z - \omega^{2006}} = \sum_{j=1}^{2006} \frac{1}{z - \omega^j} = \prod_{i \neq j} (z - \omega^i)
\]

\[
= \frac{d}{dz} \left[ \frac{1}{z^{2006} + z^{2005} + \cdots + 1} \right] = \frac{2006 z^{2005} + 2005 z^{2004} + \cdots + 1}{z^{2006} + z^{2005} + \cdots + 1} = \frac{2006 z^{2007} - 2007 z^{2006} + 1}{z^{2007} - 1}.
\]

Plugging in \(z = 2\) gives \(2005 \cdot 2^{2006} + 1\), whence the answer.

9. [7] \(g\) is a twice differentiable function over the positive reals such that

\[
g(x) + 2x^3 g'(x) + x^4 g''(x) = 0 \quad \text{for all positive reals} \quad x.
\]

\[
\lim_{x \to \infty} xg(x) = 1
\]

(1)
Find the real number $α > 1$ such that $g(α) = 1/2$.

**Answer:** $\frac{6}{π}$. In the first equation, we can convert the expression $2x^3g'(x) + x^4g''(x)$ into the derivative of a product, and in fact a second derivative, by writing $y = 1/x$. Specifically,

$$
0 = g(x) + 2x^3g'(x) + x^4g''(x) = g\left(\frac{1}{y}\right) + 2y^{-3}g\left(\frac{1}{y}\right) + y^{-4}g''\left(\frac{1}{y}\right)
$$

$$
= g\left(\frac{1}{y}\right) + \frac{d}{dy}\left[-y^{-2}g\left(\frac{1}{y}\right)\right]
$$

$$
= g\left(\frac{1}{y}\right) + \frac{d^2}{dy^2}\left[g\left(\frac{1}{y}\right)\right]
$$

Thus $g\left(\frac{1}{y}\right) = c_1\cos(y) + c_2\sin(y)$ or $g(x) = c_1\cos(1/x) + c_2\sin(1/x)$. Now the second condition gives

$$
1 = \lim_{x\to\infty} c_1x + c_2\cdot \frac{\sin(1/x)}{1/x} = c_2 + \lim_{x\to\infty} c_1x
$$

It must be that $c_1 = 0$, $c_2 = 1$. Now since $0 < 1/α < 1$, the value of $α$ such that $g(α) = \sin(1/α) = 1/2$ is given by $1/α = π/6$ and so $α = 6/π$.

10. [8] Compute

$$
\int_0^\infty \frac{e^{-x}\sin(x)}{x} dx
$$

**Answer:** $\frac{π}{4}$. We can compute the integral by introducing a parameter and exchanging the order of integration:

$$
\int_0^\infty e^{-x}\left(\frac{\sin(x)}{x}\right) dx = \int_0^\infty e^{-x}\left(\int_0^1 \cos(ax) da\right) dx = \int_0^1 \left(\int_0^\infty e^{-x}\cos(ax) dx\right) da
$$

$$
= \int_0^1 \text{Re} \left[\int_0^\infty e^{-(1+ai)x} dx\right] da = \int_0^1 \text{Re} \left[\frac{e^{-(1+ai)x}}{-1+ai}\right]_{x=0}^\infty da
$$

$$
= \int_0^1 \text{Re} \left[\frac{1}{1-ai}\right] da = \int_0^1 \text{Re} \left[\frac{1+ai}{1+a^2}\right] da
$$

$$
= \int_0^1 \frac{1}{1+a^2} da = \tan^{-1}(a)\bigg|_{a=0}^{a=1} = \frac{π}{4}
$$