11th Annual Harvard-MIT Mathematics Tournament
Saturday 23 February 2008

Individual Round: Algebra Test

1. [3] Positive real numbers $x, y$ satisfy the equations $x^2 + y^2 = 1$ and $x^4 + y^4 = \frac{17}{18}$. Find $xy$.

   **Answer:** \[ \frac{1}{5} \] We have $2x^2y^2 = (x^2 + y^2)^2 - (x^4 + y^4) = \frac{1}{18}$, so $xy = \frac{1}{5}$.

2. [3] Let $f(n)$ be the number of times you have to hit the square root key on a calculator to get a number less than 2 starting from $n$. For instance, $f(2) = 1$, $f(5) = 2$. For how many $1 < m < 2008$ is $f(m)$ odd?

   **Answer:** \[ \boxed{242} \] This is $\{2^1, 2^2 \} \cup \{2^4, 2^8 \} \cup \{2^{16}, 2^{32} \} \ldots$, and $2^8 < 2008 < 2^{16}$ so we have exactly the first two intervals.

3. [4] Determine all real numbers $a$ such that the inequality $|x^2 + 2ax + 3a| \leq 2$ has exactly one solution in $x$.

   **Answer:** \[ \boxed{1, 2} \] Let $f(x) = x^2 + 2ax + 3a$. Note that $f(-3/2) = 9/4$, so the graph of $f$ is a parabola that goes through $(-3/2, 9/4)$. Then, the condition that $|x^2 + 2ax + 3a| \leq 2$ has exactly one solution means that the parabola has exactly one point in the strip $-1 \leq y \leq 1$, which is possible if and only if the parabola is tangent to $y = 1$. That is, $x^2 + 2ax + 3a = 2$ has exactly one solution. Then, the discriminant $\Delta = 4a^2 - 4(3a - 2) = 4a^2 - 12a + 8$ must be zero. Solving the equation yields $a = 1, 2$.

4. [4] The function $f$ satisfies

   \[ f(x) + f(2x + y) + 5xy = f(3x - y) + 2x^2 + 1 \]

   for all real numbers $x, y$. Determine the value of $f(10)$.

   **Answer:** \[ -49 \] Setting $x = 10$ and $y = 5$ gives $f(10) + f(25) + 250 = f(25) + 200 + 1$, from which we get $f(10) = -49$.

   **Remark:** By setting $y = \frac{7}{8}$, we see that the function is $f(x) = -\frac{1}{2}x^2 + 1$, and it can be checked that this function indeed satisfies the given equation.

5. [5] Let $f(x) = x^3 + x + 1$. Suppose $g$ is a cubic polynomial such that $g(0) = -1$, and the roots of $g$ are the squares of the roots of $f$. Find $g(9)$.

   **Answer:** \[ 899 \] Let $a, b, c$ be the zeros of $f$. Then $f(x) = (x - a)(x - b)(x - c)$. Then, the roots of $g$ are $a^2, b^2, c^2$, so $g(x) = k(x - a^2)(x - b^2)(x - c^2)$ for some constant $k$. Since $abc = -f(0) = -1$, we have $k = ka^2b^2c^2 = -g(0) = 1$. Thus,

   \[ g(x^2) = (x^2 - a^2)(x^2 - b^2)(x^2 - c^2) = (x - a)(x - b)(x - c)(x + a)(x + b)(x + c) = -f(x)f(-x). \]

   Setting $x = 3$ gives $g(9) = -f(3)f(-3) = -(31)(-29) = 899$.

6. [5] A root of unity is a complex number that is a solution to $z^n = 1$ for some positive integer $n$. Determine the number of roots of unity that are also roots of $z^2 + az + b = 0$ for some integers $a$ and $b$.

   **Answer:** \[ 8 \] The only real roots of unity are 1 and $-1$. If $\zeta$ is a complex root of unity that is also a root of the equation $z^2 + az + b$, then its conjugate $\bar{\zeta}$ must also be a root. In this case, $|a| = |\zeta + \bar{\zeta}| \leq |\zeta| + |\bar{\zeta}| = 2$ and $b = \zeta \bar{\zeta} = 1$. So we only need to check the quadratics $z^2 + 2z + 1, z^2 + z + 1, z^2 + z + 1, z^2 + z + 1, z^2 - 2z + 1$. We find 8 roots of unity: $\pm 1, \pm i, \frac{1}{2} (\pm 1 \pm \sqrt{3}i)$.

7. [5] Compute \[ \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} \frac{k}{2^n k}. \]
Answer: \[
\frac{4}{9}
\] We change the order of summation:

\[
\sum_{n=1}^{\infty} \sum_{k=1}^{n-1} \frac{k}{2^n+k} = \sum_{k=1}^{\infty} k \sum_{n=k+1}^{\infty} \frac{1}{2^n} = \sum_{k=1}^{\infty} \frac{k}{4k} = \frac{4}{9}.
\]

(The last two steps involve the summation of an infinite geometric series, and what is sometimes called an infinite arithmetico-geometric series. These summations are quite standard, and thus we omit the details here.)

8. [6] Compute \(\arctan (\tan 65^\circ - 2 \tan 40^\circ)\). (Express your answer in degrees as an angle between 0° and 180°.)

Answer: \[25^\circ\]

First Solution: We have

\[
\tan 65^\circ - 2 \tan 40^\circ = \cot 25^\circ - 2 \cot 50^\circ = \cot 25^\circ - \frac{\cot^2 25^\circ - 1}{\cot 25^\circ} = \frac{1}{\cot 25^\circ} = \tan 25^\circ.
\]

Therefore, the answer is \(25^\circ\).

Second Solution: We have

\[
\tan 65^\circ - 2 \tan 40^\circ = \frac{1 + \tan 20^\circ}{1 - \tan 20^\circ} - 4 \tan 20^\circ = \frac{(1 - \tan 20^\circ)^2}{(1 - \tan 20^\circ)(1 + \tan 20^\circ)} = \tan(45^\circ - 20^\circ) = \tan 25^\circ.
\]

Again, the answer is \(25^\circ\).

9. [7] Let \(S\) be the set of points \((a, b)\) with \(0 \leq a, b \leq 1\) such that the equation

\[x^4 + ax^3 - bx^2 + ax + 1 = 0\]

has at least one real root. Determine the area of the graph of \(S\).

Answer: \[\frac{1}{4}\]

After dividing the equation by \(x^2\), we can rearrange it as

\[
\left(x + \frac{1}{x}\right)^2 + a \left(x + \frac{1}{x}\right) - b - 2 = 0.
\]

Let \(y = x + \frac{1}{x}\). We can check that the range of \(x + \frac{1}{x}\) as \(x\) varies over the nonzero reals is \((-\infty, -2] \cup [2, \infty)\). Thus, the following equation needs to have a real root:

\[y^2 + ay - b - 2 = 0.\]

Its discriminant, \(a^2 + 4(b + 2)\), is always positive since \(a, b \geq 0\). Then, the maximum absolute value of the two roots is

\[
\frac{a + \sqrt{a^2 + 4(b + 2)}}{2}.
\]

We need this value to be at least 2. This is equivalent to

\[
\sqrt{a^2 + 4(b + 2)} \geq 4 - a.
\]

We can square both sides and simplify to obtain

\[
2a \geq 2 - b.
\]

This equation defines the region inside \([0, 1] \times [0, 1]\) that is occupied by \(S\), from which we deduce that the desired area is \(1/4\).
10. [8] Evaluate the infinite sum

$$\sum_{n=0}^{\infty} \left(\binom{2n}{n}\right) \frac{1}{5^n}.$$

**Answer:** $\sqrt{5}$

**First Solution:** Note that

$$\binom{2n}{n} = \frac{(2n)!}{n! \cdot n!} = \frac{(2n)(2n-2)(2n-4) \cdots (2)}{n!} \cdot \frac{(2n-1)(2n-3)(2n-5) \cdots (1)}{n!}$$

$$= 2^n \cdot \frac{(-2)^n}{n!} \left(-\frac{1}{2}\right) \left(-\frac{1}{2} - 1\right) \left(-\frac{1}{2} - 2\right) \cdots \left(-\frac{1}{2} - n + 1\right)$$

$$= (-4)^n \left(-\frac{1}{2}\right)^n.$$

Then, by the binomial theorem, for any real $x$ with $|x| < \frac{1}{4}$, we have

$$(1 - 4x)^{-1/2} = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (-4x)^n = \sum_{n=0}^{\infty} \left(\binom{2n}{n}\right) x^n.$$

Therefore,

$$\sum_{n=0}^{\infty} \left(\binom{2n}{n}\right) \left(\frac{1}{5}\right)^n = \frac{1}{\sqrt{1 - \frac{4}{5}}} = \sqrt{5}.$$

**Second Solution:** Consider the generating function

$$f(x) = \sum_{n=0}^{\infty} \left(\binom{2n}{n}\right) x^n.$$

It has formal integral given by

$$g(x) = I(f(x)) = \sum_{n=0}^{\infty} \frac{1}{n+1} \left(\binom{2n}{n}\right) x^{n+1} = \sum_{n=0}^{\infty} C_n x^{n+1} = x \sum_{n=0}^{\infty} C_n x^n,$$

where $C_n = \frac{1}{n+1} \left(\binom{2n}{n}\right)$ is the $n$th Catalan number. Let $h(x) = \sum_{n=0}^{\infty} C_n x^n$; it suffices to compute this generating function. Note that

$$1 + xh(x)^2 = 1 + x \sum_{i,j \geq 0} C_i C_j x^{i+j} = 1 + x \sum_{k \geq 0} \left(\sum_{i=0}^{k} C_i C_{k-i}\right) x^k = 1 + \sum_{k \geq 1} C_{k+1} x^{k+1} = h(x),$$

where we’ve used the recurrence relation for the Catalan numbers. We now solve for $h(x)$ with the quadratic equation to obtain

$$h(x) = \frac{1/x \pm \sqrt{1/x^2 - 4/x}}{2} = \frac{1 \pm \sqrt{1 - 4x}}{2x}.$$

Note that we must choose the $-$ sign in the $\pm$, since the $+$ would lead to a leading term of $\frac{1}{2}$ for $h$ (by expanding $\sqrt{1-4x}$ into a power series). Therefore, we see that

$$f(x) = D(g(x)) = D(xh(x)) = D \left(1 - \frac{1}{2} \sqrt{1 - 4x}\right) = \frac{1}{\sqrt{1 - 4x}},$$

and our answer is hence $f(1/5) = \sqrt{5}$. 

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