For real numbers $x$ and $y$, let us consider the two operations $\oplus$ and $\odot$ defined by

$$x \oplus y = \min(x, y) \quad \text{and} \quad x \odot y = x + y.$$ 

We also include $\infty$ in our set, and it satisfies $x \oplus \infty = x$ and $x \odot \infty = \infty$ for all $x$. When unspecified, $\odot$ precedes $\oplus$ in the order of operations.

1. \[10\] (Distributive law) Prove that $(x \oplus y) \odot z = x \odot z \oplus y \odot z$ for all $x, y, z \in \mathbb{R} \cup \{\infty\}$.

**Solution:** This is equivalent to proving that

$$\min(x, y) + z = \min(x + z, y + z).$$

Consider two cases. If $x \leq y$, then $LHS = x + z$ and $RHS = x + z$. If $x > y$, then $LHS = y + z$ and $RHS = y + z$. It follows that $LHS = RHS$.

2. \[10\] (Freshman’s Dream) Let $z^n$ denote $z \odot z \odot z \odot \cdots \odot z$ with $z$ appearing $n$ times. Prove that $(x \oplus y)^n = x^n \oplus y^n$ for all $x, y \in \mathbb{R} \cup \{\infty\}$ and positive integer $n$.

**Solution:** Without loss of generality, suppose that $x \leq y$, then $LHS = \min(x, y)^n = x^n = nx$, and $RHS = \min(x^n, y^n) = \min(nx, ny) = nx$.

3. \[35\] By a tropical polynomial we mean a function of the form

$$p(x) = a_n \odot x^n \oplus a_{n-1} \odot x^{n-1} \oplus \cdots \oplus a_1 \odot x \oplus a_0,$$

where exponentiation is as defined in the previous problem.

Let $p$ be a tropical polynomial. Prove that

$$p\left(\frac{x + y}{2}\right) \geq \frac{p(x) + p(y)}{2}$$

for all $x, y \in \mathbb{R} \cup \{\infty\}$. (This means that all tropical polynomials are concave.)

**Solution:** First, note that for any $x_1, \ldots, x_n, y_1, \ldots, y_n$, we have

$$\min\{x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n\} \geq \min\{x_1, x_2, \ldots, x_n\} + \min\{y_1, y_2, \ldots, y_n\}.$$ 

Indeed, suppose that $x_m + y_m = \min_i\{x_i + y_i\}$, then $x_m \geq \min_i x_i$ and $y_m \geq \min_i y_i$, and so

$$\min_i\{x_i + y_i\} = x_m + y_m \geq \min_i x_i + \min_i y_i.$$ 

Now, let us write a tropical polynomial in a more familiar notation. We have

$$p(x) = \min_{0 \leq k \leq n} \{a_k + kx\}.$$
So

\[
p\left(\frac{x+y}{2}\right) = \min_{0 \leq k \leq n} \left\{ a_k + k \left(\frac{x+y}{2}\right) \right\} \\
= \frac{1}{2} \min_{0 \leq k \leq n} \left\{ (a_k + kx) + (a_k + ky) \right\} \\
\geq \frac{1}{2} \left( \min_{0 \leq k \leq n} \{a_k + kx\} + \min_{0 \leq k \leq n} \{a_k + ky\} \right) \\
= \frac{1}{2} \left( p(x) + p(y) \right).
\]

4. \textbf{[40] (Fundamental Theorem of Algebra)} Let \( p \) be a tropical polynomial:

\[
p(x) = a_n \circ x^n \oplus a_{n-1} \circ x^{n-1} \oplus \cdots \oplus a_1 \circ x \oplus a_0, \quad a_n \neq \infty
\]

Prove that we can find \( r_1, r_2, \ldots, r_n \in \mathbb{R} \cup \{\infty\} \) so that

\[
p(x) = a_n \circ (x \oplus r_1) \circ (x \oplus r_2) \circ \cdots \circ (x \oplus r_n)
\]

for all \( x \).

\textbf{Solution:} Again, we have

\[
p(x) = \min_{0 \leq k \leq n} \{a_k + kx\}.
\]

So the graph of \( y = p(x) \) can be drawn as follows: first, draw all the lines \( y = a_k + kx \), \( k = 0, 1, \ldots, n \), then trace out the lowest broken line, which then is the graph of \( y = p(x) \).

So \( p(x) \) is piecewise linear and continuous, and has slopes from the set \( \{0, 1, 2, \ldots, n\} \). We know from the previous problem that \( p(x) \) is concave, and so its slope must be decreasing (this can also be observed simply from the drawing of the graph of \( y = p(x) \)). Then, let \( r_k \) denote the \( x \)-coordinate of the leftmost kink such that the slope of the graph is less than \( k \) to the right of this kink. Then, \( r_n \leq r_{n-1} \leq \cdots \leq r_1 \), and for \( r_{k-1} \leq x \leq r_k \), the graph of \( p \) is linear with slope \( k \). Note that is if possible that \( r_{k-1} = r_k \), if no segment of \( p \) has slope \( k \). Also, since \( a_n \neq \infty \), the leftmost piece of \( p(x) \) must have slope \( n \), and thus \( r_n \) exists, and thus all \( r_i \) exist.

Now, compare \( p(x) \) with

\[
q(x) = a_n \circ (x \oplus r_1) \circ (x \oplus r_2) \circ \cdots \circ (x \oplus r_n)
= a_n + \min(x, r_1) + \min(x, r_2) + \cdots + \min(x, r_n).
\]

For \( r_{k-1} \leq x \leq r_k \), the slope of \( q(x) \) is \( k \), and for \( x \leq r_n \) the slope of \( q \) is \( -n \) and for \( x \geq r_1 \) the slope of \( q \) is \( 0 \). So \( q \) is piecewise linear, and of course it is continuous. It follows that the graph of \( q \) coincides with that of \( p \) up to a translation. By taking any \( x < r_n \), we see that \( q(x) = a_n + nx = p(x) \), we see that the graphs of \( p \) and \( q \) coincide, and thus they must be the same function.
A *juggling sequence of length* $n$ is a sequence $j(\cdot)$ of $n$ nonnegative integers, usually written as a string

$$j(0)j(1)\ldots j(n-1)$$

such that the mapping $f : \mathbb{Z} \to \mathbb{Z}$ defined by

$$f(t) = t + j(\bar{t})$$

is a permutation of the integers. Here $\bar{t}$ denotes the remainder of $t$ when divided by $n$. In this case, we say that $f$ is the corresponding *juggling pattern*.

For a juggling pattern $f$ (or its corresponding juggling sequence), we say that it has $b$ *balls* if the permutation induces $b$ infinite orbits on the set of integers. Equivalently, $b$ is the maximum number such that we can find a set of $b$ integers $\{t_1, t_2, \ldots, t_b\}$ so that the sets $\{t_i, f(t_i), f(f(t_i)), f(f(f(t_i))), \ldots\}$ are all infinite and mutually disjoint (i.e. non-overlapping) for $i = 1, 2, \ldots, b$. (This definition will become clear in a second.)

Now is probably a good time to pause and think about what all this has to do with juggling. Imagine that we are juggling a number of balls, and at time $t$, we toss a ball from our hand up to a height $j(\bar{t})$. This ball stays up in the air for $j(\bar{t})$ units of time, so that it comes back to our hand at time $f(t) = t + j(\bar{t})$. Then, the juggling pattern presents a simplified model of how balls are juggled (for instance, we ignore information such as which hand we use to toss the ball). A throw height of 0 (i.e., $j(\bar{t}) = 0$ and $f(t) = t$) represents that no thrown takes place at time $t$, which could correspond to an empty hand. Then, $b$ is simply the minimum number of balls needed to carry out the juggling.

The following graphical representation may be helpful to you. On a horizontal line, an curve is drawn from $t$ to $f(t)$. For instance, the following diagram depicts the juggling sequence 441 (or the juggling sequences 414 and 144). Then $b$ is simply the number of contiguous “paths” drawn, which is 3 in this case.

![Juggling diagram of 441.](image)

5. \[10\] Prove that 572 is not a juggling sequence.

**Solution:** We are given $j(0) = 5$, $j(1) = 7$ and $j(2) = 2$. So $f(3) = 3 + j(0) = 8$ and $f(1) = 1 + j(1) = 8$. Thus $f(3) = f(1)$ and so $f$ is not a permutation of $\mathbb{Z}$, and hence 572 is not a juggling pattern. (In other words, there is a “collision” at times $t \equiv 2 \pmod{3}$.)
6. Suppose that \( j(0)j(1) \cdots j(n-1) \) is a valid juggling sequence. For \( i = 0, 1, \ldots, n-1 \), let \( a_i \) denote the remainder of \( j(i) + i \) when divided by \( n \). Prove that \( (a_0, a_1, \ldots, a_{n-1}) \) is a permutation of \( (0, 1, \ldots, n-1) \).

**Solution:** Suppose that \( a_i = j(i) + i - b_i n \), where \( b_i \) is an integer. Note that \( f(i - b_i n) = i - b_i n + j(i) = a_i \). Since \( \{ i - b_i n \mid i = 0, 1, \ldots, n-1 \} \) contains \( n \) distinct integers (as their residues mod \( n \) are all distinct), and \( f \) is a permutation, we see that after applying the map \( f \), the resulting set \( \{ a_0, a_1, \ldots, a_{n-1} \} \) is a set of \( n \) distinct integers. Since \( 0 \leq a_i < n \) from definition, we see that \( (a_0, a_1, \ldots, a_{n-1}) \) is a permutation of \( (0, 1, \ldots, n-1) \).

7. Determine the number of juggling sequences of length \( n \) with exactly 1 ball.

**Answer:** \( 2^n - 1 \). **Solution:** With 1 ball, we simply need to decide at times should the ball land in our hand. That is, we need to choose a non-empty subset of \( \{ 0, 1, 2, \ldots, n-1 \} \) where the ball lands. It follows that the answer is \( 2^n - 1 \).

8. Prove that the number of balls \( b \) in a juggling sequence \( j(0)j(1) \cdots j(n-1) \) is simply the average \( b = \frac{j(0) + j(1) + \cdots + j(n-1)}{n} \).

**Solution:** Consider the corresponding juggling diagram. Say the length of an curve from \( t \) to \( f(t) \) is \( f(t) - t \). Let us draw only the curves whose left endpoint lies inside \( [0, Mn - 1] \). For every single ball, the sum of the lengths of the arrows drawn corresponding to that ball is between \( Mn - J \) and \( Mn + J \), where \( J = \max \{ j(0), j(1), \ldots, j(n-1) \} \). It follows that the sum of the lengths of the arrows drawn is between \( b(Mn - J) \) and \( b(Mn + J) \). Since the arrow drawn at \( t \) has length \( j(T) \), the sum of the lengths of the arrows drawn is \( M(j(0) + j(1) + \cdots + j(n-1)) \). It follows that

\[
b(Mn - J) \leq M(j(0) + j(1) + \cdots + j(n-1)) \leq b(Mn + J).
\]

Dividing by \( Mn \), we get

\[
b \left( 1 - \frac{J}{nM} \right) \leq \frac{j(0) + j(1) + \cdots + j(n-1)}{n} \leq b \left( 1 + \frac{J}{nM} \right).
\]

Since we can take \( M \) to be arbitrarily large, we must have

\[
b = \frac{j(0) + j(1) + \cdots + j(n-1)}{n},
\]

as desired.

9. Show that the converse of the previous statement is false by providing a non-juggling sequence \( j(0)j(1)j(2) \) of length 3 where the average \( \frac{1}{3}(j(0) + j(1) + j(2)) \) is an integer. Show that your example works.

**Solution:** One such example is 210. This is not a juggling sequence since \( f(0) = f(1) = 2 \).
Incircles \[180\]

In the following problems, \(ABC\) is a triangle with incenter \(I\). Let \(D, E, F\) denote the points where the incircle of \(ABC\) touches sides \(BC, CA, AB\), respectively.

At the end of this section you can find some terminology and theorems that may be helpful to you.

10. \([15]\) Let \(a, b, c\) denote the side lengths of \(BC, CA, AB\). Find the lengths of \(AE, BF, CD\) in terms of \(a, b, c\).

**Solution:** Let \(x = AE = AF\), \(y = BD = BF\), \(z = CD = CE\). Since \(BC = BD + CD\), we have \(a = x + y\). Similarly with the other sides, we arrive at the following system of equations:

\[
\begin{align*}
a &= y + z, \\ b &= x + z, \\ c &= x + y.
\end{align*}
\]

Solving this system gives us

\[
\begin{align*}
AE &= x = \frac{b + c - a}{2}, \\
BF &= y = \frac{a + c - b}{2}, \\
CD &= z = \frac{a + b - c}{2}.
\end{align*}
\]

11. \([15]\) Show that lines \(AD, BE, CF\) pass through a common point.

**Solution:** Using Ceva’s theorem on triangle \(ABC\), we see that it suffices to show that

\[
\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.
\]

Since \(AF = AE, BD = BF\), and \(CD = CE\) (due to equal tangents), we see that the LHS is indeed 1.

*Remark:* The point of concurrency is known as the *Gergonne point*.

12. \([35]\) Show that the incenter of triangle \(AEF\) lies on the incircle of \(ABC\).

**Solution:** Let segment \(AI\) meet the incircle at \(A_1\). Let us show that \(A_1\) is the incenter of \(AEF\).
Since $AE = AF$ and $AA'$ is the angle bisector of $\angle EAF$, we find that $A_1E = A_1F$. Using tangent-chord, we see that $\angle AFA_1 = \angle A_1EF = \angle A_1FE$. Therefore, $A_1$ lies on the angle bisector of $\angle AFE$. Since $A_1$ also lies on the angle bisector of $\angle EAF$, $A_1$ must be the incenter of $AEF$, as desired.

13. [35] Let $A_1, B_1, C_1$ be the incenters of triangle $AEF, BDF, CDE$, respectively. Show that $A_1D, B_1E, C_1F$ all pass through the orthocenter of $A_1B_1C_1$.

**Solution:** Using the result from the previous problem, we see that $A_1, B_1, C_1$ are respectively the midpoints of the arc $FE, FD, DF$ of the incircle. We have

\[
\angle DA_1C_1 + \angle B_1C_1A_1 = \frac{1}{2} \angle DIC_1 + \frac{1}{2} \angle B_1IF + \frac{1}{2} \angle FIA_1 = \frac{1}{4} (\angle EID + \angle DIF + \angle FIE) = \frac{1}{4} \cdot 360^\circ = 90^\circ.
\]

It follows that $A_1D$ is perpendicular to $B_1C_1$, and thus $A_1D$ passes through the orthocenter of $A_1B_1C_1$. Similarly, $A_1D, B_1E, C_1F$ all pass through the orthocenter of $A_1B_1C_1$.

14. [40] Let $X$ be the point on side $BC$ such that $BX = CD$. Show that the excircle $ABC$ opposite of vertex $A$ touches segment $BC$ at $X$.

**Solution:** Let the excircle touch lines $BC, AC$ and $AB$ at $X', Y$ and $Z$, respectively. Using the equal tangent property repeatedly, we have

\[
BX' - X'C = BZ - CY = (EY - CY) - (FZ - BZ) = CE - BF = CD - BD.
\]

It follows that $BX' = CD$, and thus $X' = X$. So the excircle touches $BC$ at $X$. 

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15. Let $X$ be as in the previous problem. Let $T$ be the point diametrically opposite to $D$ on the incircle of $ABC$. Show that $A, T, X$ are collinear.

**Solution:** Consider a dilation centered at $A$ that carries the incircle to the excircle. This dilation must send the diameter $DT$ to some the diameter of excircle that is perpendicular to $BC$. The only such diameter is the one goes through $X$. It follows that $T$ gets carried to $X$. Therefore, $A, T, X$ are collinear.

**Glossary and some possibly useful facts**

- A set of points is *collinear* if they lie on a common line. A set of lines is *concurrent* if they pass through a common point.

- Given $ABC$ a triangle, the three angle bisectors are concurrent at the *incenter* of the triangle. The incenter is the center of the *incircle*, which is the unique circle inscribed in $ABC$, tangent to all three sides.

- The *excircles* of a triangle $ABC$ are the three circles on the exterior the triangle but tangent to all three lines $AB, BC, CA$.

- The *orthocenter* of a triangle is the point of concurrency of the three altitudes.
• Ceva’s theorem states that given $ABC$ a triangle, and points $X, Y, Z$ on sides $BC, CA, AB$, respectively, the lines $AX, BY, CZ$ are concurrent if and only if

$$\frac{BX}{XB} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$