HMMT 2013
Saturday 16 February 2013
Team Round

1. [10] Let $a$ and $b$ be real numbers such that $\frac{ab}{a^2 + b^2} = \frac{1}{4}$. Find all possible values of $\frac{|a^2 - b^2|}{a^2 + b^2}$.

**Answer:** $\frac{\sqrt{3}}{2}$ The hypothesis statement is equivalent to

\[
a^2 + b^2 = 4ab
1 : (a + b)^2 = 6ab
2 : (a - b)^2 = 2ab
\]

Multiplying equations 1 and 2,

\[
(a^2 - b^2)^2 = 12(ab)^2
|a^2 - b^2| = \pm \sqrt{12}ab
\]

Since the left hand side and $ab$ are both positive,

\[
|a^2 - b^2| = \sqrt{12}ab
\]

\[
\frac{|a^2 - b^2|}{a^2 + b^2} = \frac{\sqrt{12}ab}{a^2 + b^2} = \frac{\sqrt{12}}{4} = \frac{\sqrt{3}}{2}
\]

(It is clear that such $a$ and $b$ exist: for example, we can take $a = 1$ and solve for $b$ by way of the quadratic formula.)

2. [15] A cafe has 3 tables and 5 individual counter seats. People enter in groups of size between 1 and 4, inclusive, and groups never share a table. A group of more than 1 will always try to sit at a table, but will sit in counter seats if no tables are available. Conversely, a group of 1 will always try to sit at the counter first. One morning, $M$ groups consisting of a total of $N$ people enter and sit down. Then, a single person walks in, and realizes that all the tables and counter seats are occupied by some person or group. What is the minimum possible value of $M + N$?

**Answer:** 16 We first show that $M + N \geq 16$. Consider the point right before the last table is occupied. We have two cases: first suppose there exists at least one open counter seat. Then, every table must contribute at least 3 to the value of $M + N$, because no groups of 1 will have taken a table with one of the counter seats open. By the end, the counter must contribute at least 5 + 2 = 7 to $M + N$, as there must be at least 2 groups sitting at the counter. It follows that $M + N \geq 16$. For the second case, assume the counter is full right before the last table is taken. Then, everybody sitting at the counter must have entered as a singleton, since they entered when a table was still available. Consequently, the counter must contribute 10 to $M + N$, and each table contributes at least 2, so once again $M + N \geq 16$.

Now, $M + N = 16$ is achievable with eight groups of one, who first fill the counter seats, then the three tables. Thus, our answer is 16.

3. [15] Let $ABC$ be a triangle with circumcenter $O$ such that $AC = 7$. Suppose that the circumcircle of $AOC$ is tangent to $BC$ at $C$ and intersects the line $AB$ at $A$ and $F$. Let $FO$ intersect $BC$ at $E$. Compute $BE$.

**Answer:** $\frac{EB}{2}$ $O$ is the circumcenter of $\triangle ABC \implies AO = CO \implies \angle OCA = \angle OAC$.

Because $AC$ is an inscribed arc of circumcircle $\triangle AOC$, $\angle OCA = \angle OFA$. Furthermore $BC$ is tangent to circumcircle $\triangle AOC$, so $\angle OAC = \angle OCB$. However, again using the fact that $O$ is the circumcenter of $\triangle ABC$, $\angle OCB = \angle OBC$.

We now have that $CO$ bisects $\angle ACB$, so it follows that triangle $CA = CB$. Also, by AA similarity we have $EOB \sim EBF$. Thus, $EB^2 = EO \cdot EF = EC^2$ by the similarity and power of a point, so $EB = BC/2 = AC/2 = 7/2$.
4. [20] Let $a_1, a_2, a_3, a_4, a_5$ be real numbers whose sum is 20. Determine with proof the smallest possible value of

$$\sum_{1 \leq i < j \leq 5} \lfloor a_i + a_j \rfloor.$$ 

**Answer:** We claim that the minimum is 72. This can be achieved by taking $a_1 = a_2 = a_3 = a_4 = 0.4$ and $a_5 = 18.4$. To prove that this is optimal, note that

$$\sum_{1 \leq i < j \leq 5} (a_i + a_j) = \sum_{1 \leq i < j \leq 5} \{a_i + a_j\} = 80 - \sum_{1 \leq i < j \leq 5} \lfloor a_i + a_j \rfloor,$$

so it suffices to maximize

$$\sum_{1 \leq i < j \leq 5} \lfloor a_i + a_j \rfloor = \sum_{i=1}^{5} \lfloor a_i + a_{i+2} \rfloor + \sum_{i=1}^{5} \lfloor a_i + a_{i+1} \rfloor,$$

where $a_6 = a_1$ and $a_7 = a_2$. Taking each sum modulo 1, it is clear that both are integers. Thus, the above sum is at most $2 \cdot 4 = 8$, and our original expression is at least $80 - 8 = 72$, completing the proof.

5. [25] Thaddeus is given a $2013 \times 2013$ array of integers each between 1 and 2013, inclusive. He is allowed two operations:

1. Choose a row, and subtract 1 from each entry.
2. Choose a column, and add 1 to each entry.

He would like to get an array where all integers are divisible by 2013. On how many arrays is this possible?

**Answer:** We claim that the set of grids on which it is possible to obtain an array of all zeroes (mod 2013) is indexed by ordered 4025-tuples of residues (mod 2013), corresponding to the starting entries in the first row and first column of the grid, giving the answer of 2013. To do this, we show that given after fixing all of the entries in the first row and column, there is a unique starting grid which can become an array of all zeroes after applying the appropriate operations.

Let $a_{i,j}$ be the entry in the $i$-th row and the $j$-th column. Suppose there is a sequence of operations giving all zeroes in the array; let $r_i$ be the number of times we operate on row $i$, and let $c_j$ be the number of times we operate on column $j$. It is enough to take all of these values to be residues modulo 2013. Clearly, $a_{i,j} + r_i + c_j \equiv 0 \pmod{2013}$ for each $i,j$. In particular, $r_1 + c_1 \equiv a_{1,1}$. Now, for each $i,j$, we have

$$a_{i,j} \equiv -r_i - c_j \equiv (a_{i,1} + c_1) + (a_{1,j} + r_1) \equiv a_{i,1} + a_{1,j} - a_{1,1},$$

which is fixed. Thus, there rest of the entries in the grid are forced.

Conversely, if we set $a_{i,j}$ to be the appropriate representative of the residue class of $a_{i,1} + a_{1,j} - a_{1,1}$ modulo 2013, we may take $r_i \equiv -a_{i,1} \pmod{2013}$, and $c_j \equiv a_{1,1} - a_{1,j} \pmod{2013}$ for each $i,j$. It is clear that $a_{i,j} + r_i + c_j \equiv 0 \pmod{2013}$ for each $i,j$, so we’re done.

6. [25] Let triangle $ABC$ satisfy $2BC = AB + AC$ and have incenter $I$ and circumcircle $\omega$. Let $D$ be the intersection of $AI$ and $\omega$ (with $A,D$ distinct). Prove that $I$ is the midpoint of $AD$.

**Answer:** Since $AD$ is an angle bisector, $D$ is the midpoint of arc $BC$ opposite $A$ on $\omega$. It is well-known that $B$, $I$, and $C$ lie on a circle centered at $D$. Thus $BD = DC = DI$. Applying Ptolemy’s theorem to cyclic quadrilateral $ABDC$, we get

$$AB \cdot DC + AC \cdot BD = AD \cdot BC = AD \cdot (AB + AC)/2.$$
Using \( BD = DC \) we have immediately that \( AD = 2BD = 2DI \) so \( I \) is the midpoint of \( AD \) as desired.

**Solution 2:** Let \( P \) and \( Q \) be the midpoints of \( AB \) and \( AC \), and take the point \( E \) on segment \( BC \) such that \( BE = BP \). Note that \( CE = AB - BE = AB - BP = \frac{AB + AC - AB}{2} = \frac{AC}{2} = CQ \), so triangles \( BPE \) and \( CQE \) are isosceles. In addition, \( \frac{BE}{EC} = \frac{AB/2}{AC/2} = \frac{AB}{AC} \), so by the angle bisector theorem, \( AE \) bisects \( \angle CAB \), whence \( E \) must lie on the bisector of \( \angle A \).

Since triangles \( BPE \) and \( CQE \) are isosceles, the bisectors of angles \( B \) and \( C \) are the perpendicular bisectors of segments \( PE \) and \( EQ \), respectively. Thus, the circumcenter of \( \triangle PQE \) is \( I \), so the perpendicular bisector of \( PQ \) meets the bisector of \( \angle A \) at \( I \).

Furthermore, since \( \angle DAB = \angle CAD \), arcs \( DB \) and \( DC \) have the same measure, so \( BD = DC \), whence the perpendicular bisector of \( BC \) meets the bisector of \( \angle A \) at \( D \). A homothety centered at \( A \) with factor \( 1/2 \) maps \( BC \) to \( PQ \), and so maps \( D \) to \( I \). Thus, \( D \) is the midpoint of \( AI \).

**Solution 3:** Let \( a = BC \), \( b = CA \), and \( c = AB \), let \( r \) and \( R \) denote the lengths of the inradius and circumradius of \( \triangle ABC \), respectively, let \( E \) be the intersection of segments \( AD \) and \( BC \), and let \( O \) be the circumcenter of \( \triangle ABC \). It is easy to see that \( OI \) is a chord if and only if \( OI \perp AD \), which is true if and only if \( OA^2 = OF^2 + AI^2 \). By Euler’s distance formula, \( OI^2 = R(2 - 2r) \), and by Stewart’s theorem and the angle bisector theorem we can find \( AI^2 = \left( \frac{a+b+c}{2} \right)^2 \). Thus, it remains to show that \( 6Rr = bc \).

Now, we use the combine the well-known formulas \( \frac{abc}{4K} = R \) and \( K = rs \) to get \( abc = 4Rrs \), where \( K \) is the area of \( \triangle ABC \) and \( s \) is its semiperimeter. We have \( bc = Rr \frac{a}{a} = Rr \frac{2a+2b+c}{a} = Rr \frac{2a+2b+2c}{a} = 6Rr \), as desired.

7. [30] There are \( n \) children and \( n \) toys such that each child has a strict preference ordering on the toys. We want to distribute the toys: say a distribution \( A \) dominates a distribution \( B \) if in \( A \), each child receives at least as preferable of a toy as in \( B \). Prove that if some distribution is not dominated by any other, then at least one child gets his/her favorite toy in that distribution.

**Answer:** \( N/A \) Suppose we have a distribution \( A \) assigning each child \( C_i \), \( i = 1, 2, \ldots, n \), toy \( T_i \), such that no child \( C_i \) gets their top preference \( T_i \). Then, pick an arbitrary child \( C_1 \) and construct the sequence of children \( C_1, C_2, C_3, \ldots \) where \( i = 1 \) and \( C_{i+1} \) was assigned the favorite toy \( T'_i \) of the last child \( C_i \). Eventually, some \( C_{i_k} = C_i \); at this point, just pass the toys around this cycle so that each of these children gets their favorite toy. Clearly the resulting distribution dominates the original, so we’re done.

8. [35] Let points \( A \) and \( B \) be on circle \( \omega \) centered at \( O \). Suppose that \( \omega_A \) and \( \omega_B \) are circles not containing \( O \) which are internally tangent to \( \omega \) at \( A \) and \( B \), respectively. Let \( \omega_A \) and \( \omega_B \) intersect at \( C \) and \( D \) such that \( D \) is inside triangle \( ABC \). Suppose that line \( BC \) meets \( \omega \) again at \( E \) and let line \( EA \) intersect \( \omega_A \) at \( F \). If \( FC \perp CD \), prove that \( O \), \( C \), and \( D \) are collinear.

**Answer:** \( N/A \) Let \( H = CA \cap \omega \), and \( G = BH \cap \omega_B \). There are homotheties centered at \( A \) and \( B \) taking \( \omega_A \to \omega \) and \( \omega_B \to \omega \) that take \( A : F \leftrightarrow E, A : C \leftrightarrow H, B : C \leftrightarrow E \) and \( B : G \leftrightarrow H \). In particular \( CF \parallel EH \parallel CG \), so \( C, F, G \) are collinear, lying on a line perpendicular to \( DC \).

Because of the right angles at \( C \), \( DF \) and \( DG \) are diameters of \( \omega_A, \omega_B \), respectively. Also, we have that the ratio of the sizes of \( \omega_A \) and \( \omega_B \), under the two homotheties above, is \( CF/CH \cdot EH/CG = CF/CG \). Therefore, \( DF/DG = CF/CG \), but then \( \triangle DCF \) and \( \triangle DCG \) are both right triangles which share one side and have hypotenuse and other side in proportion; it is obvious now that the two circles \( \omega_A \) and \( \omega_B \) are congruent.

Therefore, \( O \) has the same distance to \( A \) and \( B \), and so the same distances to the centers of the two circles as well. As a result, \( O \) lies on the radical axis \( CD \) as desired.

9. [35] Let \( m \) be an odd positive integer greater than 1. Let \( S_m \) be the set of all non-negative integers less than \( m \) which are of the form \( x + y \), where \( xy - 1 \) is divisible by \( m \). Let \( f(m) \) be the number of elements of \( S_m \).
(a) Prove that $f(mn) = f(m)f(n)$ if $m, n$ are relatively prime odd integers greater than 1.

(b) Find a closed form for $f(p^k)$, where $k > 0$ is an integer and $p$ is an odd prime.

Answer: [N/A] For a positive integer $n$, let $\mathbb{Z}/n\mathbb{Z}$ denote the set of residues modulo $n$ and $(\mathbb{Z}/n\mathbb{Z})^*$ denote the set of residues modulo $n$ that are relatively prime to $n$. Then, rephrased, $S_m$ is the set of residues modulo $m$ of the form $x + x^{-1}$, where $x \in (\mathbb{Z}/m\mathbb{Z})^*$.

For part (a), suppose $a = x + x^{-1} \in S_m$ and $b = y + y^{-1} \in S_n$. By the Chinese Remainder Theorem, there exists a residue $z \in (\mathbb{Z}/mn\mathbb{Z})^*$ such that $m|(x - z)$ and $n|(y - z)$, and thus $z + z^{-1} \equiv x + x^{-1}$ (mod $m$) and $z + z^{-1} \equiv y + y^{-1}$ (mod $n$). Therefore, all $f(m)f(n)$ residues modulo $mn$ which result from applying the Chinese Remainder Theorem to each element of $S_m$ and $S_n$ are in $S_{mn}$. Conversely, given $z + z^{-1} \in (\mathbb{Z}/mn\mathbb{Z})^*$, taking $z + z^{-1}$ modulo $m$ and $n$ gives elements of $S_m$ and $S_n$, so indeed $f(mn) = f(m)f(n)$.

We now proceed to part (b). For each $x \in (\mathbb{Z}/p^k\mathbb{Z})^*$, denote $q(x)$ to be the largest non-negative integer $i \leq k$ such that $p^i$ divides $x^2 - 1$ (this is clearly well-defined). For a given $x$, let $g(x)$ be the number of $y \in (\mathbb{Z}/p^k\mathbb{Z})^*$ such that $x + x^{-1} \equiv y + y^{-1}$ (mod $p^k$). Note that this condition is equivalent to $(x - y)(xy - 1) \equiv 0$ (mod $p^k$).

First, consider the case in which $q(x) \geq k/2$, in which case we have $p^{[k/2]}|(x - 1)(x + 1)$, and because $p$ is odd, $x \equiv \pm 1$ (mod $p^{[k/2]}$). Thus, either $(x - 1)^2 \equiv 0$ (mod $p^k$) or $(x + 1)^2 \equiv 0$ (mod $p^k$), and it follows that $x + x^{-1} \equiv \pm 2$ (mod $p^k$) (clearly 2 and $-2$ are distinct). Conversely, $x + x^{-1} \equiv \pm 2$ implies $(x \pm 1) \equiv 0$ (mod $p^{[k/2]}$), which in turn implies $q(x) \geq k/2$. It is now clear that there are exactly two elements of $S_m$ corresponding to residues of the form $x + x^{-1}$ with $q(x) \geq k/2$, and all other elements of $S_m$ come from $x$ with $q(x) < k/2$.

Fix $x$ with $q(x) < k/2$; we will compute $g(x)$. Suppose $(x - y)(xy - 1) \equiv 0$ (mod $p^k$), and say $x - y, xy - 1$ have $j, j'$ factors of $p$, respectively. If $j \leq j'$, note that $x^2 - 1 \equiv xy - 1 \equiv 0$ (mod $p^j$), so $j \leq q(x)$. Similarly, if $j' \leq j$, $x \equiv y$ (mod $p^{j'}$), so $x^2 - 1 \equiv xy - 1 \equiv 0$ (mod $p^j$), and so $j' \leq q(x)$. It follows that $\min(j, j') \leq q(x)$, and thus $\max(j, j') \geq k - q(x)$.

Suppose $p^{k - q(x)}|(x - y)$. Then, $xy - 1 \equiv x^2 - 1 \equiv 0$ (mod $p^{q(x)}$) because $q(x) < k/2$, so any $y$ with $p^{k - q(x)}|(x - y)$ satisfies $(x - y)(xy - 1) \equiv 0$ (mod $p^k$). Now, suppose $p^{k - q(x)}|(xy - 1)$, that is, $y \equiv x^{-1}$ (mod $p^{k - q(x)}$). Then, $xy \equiv 1$ (mod $p^{q(x)}$), and since $x^2 \equiv 1$ (mod $p^{q(x)}$), we have $x \equiv y$ (mod $p^{q(x)}$), so again we have $(x - y)(xy - 1) \equiv 0$ (mod $p^k$). It follows that the set of $y$ satisfying $(x - y)(xy - 1) \equiv 0$ (mod $p^k$) is exactly the set of $y$ with $y \equiv x$ (mod $p^{k - q(x)}$) or $y \equiv x^{-1}$ (mod $p^{k - q(x)}$). $x, x^{-1}$ are distinct residues modulo $p^{k - q(x)}$, because $x^2 - 1$ has fewer than $k/2$ factors of $p$, so it follows that $g(x) = 2p^{q(x)}$.

In particular, note that these values are distinct for different values of $q(x) < k/2$, so $x + x^{-1} \equiv y + y^{-1}$ implies $q(x) = q(y)$ or $q(x), q(y) \geq k/2$.

For each integer $i$ with $0 \leq i < k/2$, we need to compute the number of $x \in (\mathbb{Z}/p^k\mathbb{Z})$ with $q(x) = i$. Clearly, this is the number of $x$ with $p^i|x^2 - 1$ minus the number of $x$ with $p^{i+1}|x^2 - 1$. When $i = 0$, the number of $x$ with $p^i|x^2 - 1$ is clearly $p^{k-1}(p-1)$, and when $i > 0$, this number is $2p^{k-i}$, as we have $x \equiv \pm 1$ (mod $p^i$).

We can now count the number of elements of $S_{p^k}$ using casework on the value of $q(x)$ where we take $x$ to be such that $x + x^{-1}$ is a particular element of $S_m$. Applying the results from the previous two paragraphs, our answer is

$$2 + \frac{p^{k-1}(p-3)}{2} + \sum_{i=1}^{[k/2]-1} \frac{2p^{k-i} - 2p^{k-i-1}}{2p^i},$$

where the summand 2 comes from $\pm 2 \in S_{p^k}$, corresponding to all $x$ with $q(x) > k/2$, the next summand comes from those $x$ with $q(x) = 0$, and each additional summand comes from those $x$ with $q(x) = i$ in the relevant range. We may evaluate the last sum as a geometric series, to obtain the final closed form answer of

$$2 + \frac{p^{k-1}(p-3)}{2} + \frac{p^{k-1} - p^{1+(i-1)k}}{p + 1}. $$

Team Round
Chim Tu has a large rectangular table. On it, there are finitely many pieces of paper with non-overlapping interiors, each one in the shape of a convex polygon. At each step, Chim Tu is allowed to slide one piece of paper in a straight line such that its interior does not touch any other piece of paper during the slide. Can Chim Tu always slide all the pieces of paper off the table in finitely many steps?

**Answer:** N/A

Let the pieces of paper be $P_1, P_2, \ldots, P_n$ in the Cartesian plane. It suffices to show that for any constant distance $D$, they can be slid so that each pairwise distance is at least $D$. Then, we can apply this using $D$ equal to the diameter of the rectangle, sliding all but at most one of the pieces of paper off the table, and then slide this last one off arbitrarily.

We show in particular that there is always a polygon $P_i$ which can be slid arbitrarily far to the right (i.e. in the positive $x$-direction). For each $P_i$, let $B_i$ be a bottommost (i.e. lowest $y$-coordinate) point on the boundary of $P_i$. Define a point $Q$ to be *exposed* if the ray starting at $Q$ in the positive $x$ direction meets the interior of no piece of paper.

Consider the set of all exposed $B_i$; this set is nonempty because certainly the bottommost of all the $B_i$ is exposed. Of this set, let $B_k$ be the exposed $B_i$ with maximal $y$-coordinate, and if there are more than one such, choose the one with maximal $x$-coordinate. We claim that the corresponding $P_k$ can be slid arbitrarily far to the right.

Suppose for the sake of contradiction that there is some polygon blocking this path. To be precise, if $A_k$ is the highest point of $P_k$, then the region $R$ formed by the right-side boundary of $P_k$ and the rays pointing in the positive $x$ direction from $A_k$ and $B_k$, must contain interior point(s) of some set of polygon(s) $P_j$ in its interior. All of their bottommost points $B_j$ must lie in $R$, since none of them can have boundary intersecting the ray from $B_k$, by the construction of $B_k$.

Because $B_k$ was chosen to be rightmost out of all the exposed $B_i$ with that $y$-coordinate, it must be that all of the $B_j$ corresponding to the blocking $P_j$ have larger $y$-coordinate. Now, choose of these the one with smallest $y$-coordinate - it must be exposed, and it has strictly higher $y$-coordinate than $B_k$, contradiction. It follows that the interior of $R$ intersects no pieces of paper.

Now, for a fixed $D$ such that $D$ is at least the largest distance between any two points of two polygons, we can shift this exposed piece of paper $nD$ to the right, the next one of the remaining pieces $(n-1)D$, and so on, so that the pairwise distances between pieces of paper, even when projected onto the $x$-axis, are at least $D$ each. We’re done.