1. [2] What is the smallest non-square positive integer that is the product of four prime numbers (not necessarily distinct)?

Answer: 24 The smallest two integers that are the product of four primes are $2^4 = 16$ and $2^3 \cdot 3 = 24$. Since 16 is a perfect square and 24 is not, the answer is 24.

2. [3] Plot points $A, B, C$ at coordinates $(0,0)$, $(0,1)$, and $(1,1)$ in the plane, respectively. Let $S$ denote the union of the two line segments $AB$ and $BC$. Let $X_1$ be the area swept out when Bobby rotates $S$ counterclockwise 45 degrees about point $A$. Let $X_2$ be the area swept out when Calvin rotates $S$ clockwise 45 degrees about point $A$. Find $X_1 + X_2$.

Answer: $\frac{7\sqrt{3}}{4}$ It’s easy to see $X_1 = X_2$. Simple cutting and pasting shows that $X_1$ equals the area of $\frac{1}{8}$ of a circle with radius $AC = \sqrt{2}$, so $X_1 + X_2 = X_1 = \frac{1}{8} \pi (\sqrt{2})^2 = \frac{\pi}{4}$.

3. [4] A 24-hour digital clock shows times $h : m : s$, where $h$, $m$, and $s$ are integers with $0 \leq h \leq 23$, $0 \leq m \leq 59$, and $0 \leq s \leq 59$. How many times $h : m : s$ satisfy $h + m = s$?

Answer: 1164 We are solving $h + m = s$ in $0 \leq s \leq 59$, $0 \leq m \leq 59$, and $0 \leq h \leq 23$. If $s \geq 24$, each $h$ corresponds to exactly 1 solution, so we get $24(59-23) = 24(36)$ in this case. If $s \leq 23$, we want the number of nonnegative integer solutions to $h + m \leq 23$, which by lattice point counting (or balls and urns) is $\binom{23+2}{2} = (23 + 2)(23 + 1)/2 = 25 \cdot 12$. Thus our total is $12(72 + 25) = 12(100 - 3) = 1164$.

4. [4] A 50-card deck consists of 4 cards labeled “i” for $i = 1, 2, \ldots, 12$ and 2 cards labeled “13”. If Bob randomly chooses 2 cards from the deck without replacement, what is the probability that his 2 cards have the same label?

Answer: $\frac{73}{1225}$ All pairs of distinct cards (where we distinguish cards even with the same label) are equally likely. There are $\binom{50}{2} = 1225$ pairs overall, and $\binom{14}{2} = 73$ pairs with the same label and $\binom{10}{2} = 45$ pairs with different labels. Thus the probability is $\frac{73}{1225}$.

5. [5] Let $ABC$ be an isosceles triangle with $AB = AC$. Let $D$ and $E$ be the midpoints of segments $AB$ and $AC$, respectively. Suppose that there exists a point $F$ on ray $D\overrightarrow{E}$ outside of $ABC$ such that triangle $BFA$ is similar to triangle $ABC$. Compute $\frac{AB}{BC}$.

Answer: $\frac{1}{2}$ Let $\alpha = \angle ABC = \angle ACB$, $AB = 2x$, and $BC = 2y$, so $AD = DB = AE = EC = x$ and $DE = y$. Since $\triangle BFA \sim \triangle ABC$ and $BA = AC$, we in fact have $\triangle BFA \cong \triangle ABC$, so $BF = BA = 2x$, $FA = 2y$, and $\angle DAF = \alpha$. But $DE \parallel BC$ yields $\angle ADF = \angle ABC = \alpha$ as well, whence $\triangle FAD \sim \triangle ABC$ gives $\frac{2x}{y} = \frac{FA}{AD} = \frac{AB}{BC} = \frac{2x}{2y} \implies \frac{AB}{BC} = \frac{2}{y} = \sqrt{2}$.

6. [5] Find the number of positive integer divisors of 12! that leave a remainder of 1 when divided by 3.

Answer: 66 First we factor $12! = 2^{10}3^55^27^111^1$, and note that $2, 5, 11 \equiv -1 \pmod{3}$ while $7 \equiv 1 \pmod{3}$. The desired divisors are precisely $2^a 5^b 7^c 11^d$ with $0 \leq a \leq 10, 0 \leq b \leq 2, 0 \leq c \leq 1, 0 \leq d \leq 1$, and $a + b + d$ even. But then for any choice of $a, b$, exactly one $d \in \{0, 1\}$ makes $a + b + d$ even, so we have exactly one 1 (mod 3)-divisor for every triple $(a, b, c)$ satisfying the inequality constraints. This gives a total of $(10 + 1)(2 + 1)(1 + 1) = 66$.

7. [6] Find the largest real number $\lambda$ such that $a^2 + b^2 + c^2 + d^2 \geq ab + \lambda bc + cd$ for all real numbers $a, b, c, d$.

Answer: $\frac{1}{3}$ Let $f(a, b, c, d) = (a^2 + b^2 + c^2 + d^2) - (ab + \lambda bc + cd)$. For fixed $(b, c, d)$, $f$ is minimized at $a = \frac{b}{2}$, and for fixed $(a, b, c)$, $f$ is minimized at $d = \frac{c}{2}$, so simply we want the largest $\lambda$ such that $f(\frac{1}{2}, b, c, \frac{1}{2}) = \frac{1}{4}(b^2 + c^2) - \lambda bc$ is always nonnegative. By AM-GM, this holds if and only if $\lambda \leq 2\frac{1}{4} = \frac{1}{2}$. HMMT November 2013
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General Test
8. The first 1000 positive integers can be written as the sum of finitely many distinct numbers from the sequence \(3^0, 3^1, 3^2, \ldots\)?

**Answer:** \(105\) We want to find which integers have only 0’s and 1’s in their base 3 representation. Note that \(1000_{10} = 1101001_3\). We can construct a bijection from all such numbers to the binary strings, by mapping \(x_3 \leftrightarrow x_2\). Since \(1101001_{10} = 105_{10}\), we conclude that the answer is 105.

9. Let \(ABC\) be a triangle and \(D\) a point on \(BC\) such that \(AB = \sqrt{2}, AC = \sqrt{3}, \angle BAD = 30^\circ, \angle CAD = 45^\circ\). Find \(AD\).

**Answer:** \(\frac{\sqrt{6}}{2} \text{ OR } \frac{\sqrt{3}}{\sqrt{2}}\) Note that \([BAD] + [CAD] = [ABC]\). If \(\alpha_1 = \angle BAD, \alpha_2 = \angle CAD\), then we deduce \(\frac{\sin(\alpha_1 + \alpha_2)}{AD} = \frac{\sin \alpha_1}{AC} + \frac{\sin \alpha_2}{AB}\) upon division by \(AB \cdot AC \cdot AD\). Now

\[
AD = \frac{\sin(30^\circ + 45^\circ)}{\sin 30^\circ + \sin 45^\circ}.
\]

But \(\sin(30^\circ + 45^\circ) = \sin 30^\circ \cos 45^\circ + \sin 45^\circ \cos 30^\circ = \sin 30^\circ \frac{1}{\sqrt{2}} + \sin 45^\circ \frac{\sqrt{3}}{2} = \frac{\sqrt{6}}{2} \left( \frac{\sin 30^\circ}{\sqrt{3}} + \frac{\sin 45^\circ}{\sqrt{2}} \right)\), so our answer is \(\frac{\sqrt{6}}{2}\).

10. How many functions \(f : \{1, 2, \ldots, 2013\} \rightarrow \{1, 2, \ldots, 2013\}\) satisfy \(f(j) < f(i) + j - i\) for all integers \(i, j\) such that \(1 \leq i < j \leq 2013\)?

**Answer:** \(\binom{4025}{2013}\) Note that the given condition is equivalent to \(f(j) - j < f(i) - i\) for all \(1 \leq i < j \leq 2013\). Let \(g(i) = f(i) - i\), so that the condition becomes \(g(j) < g(i)\) for \(i < j\) and \(1 - i \leq g(i) \leq 2013 - i\). However, since \(g\) is decreasing, we see by induction that \(g(i + 1)\) is in the desired range so long as \(g(i)\) is in the desired range. Hence, it suffices to choose 2013 values for \(g(1), \ldots, g(2013)\) in decreasing order from \([-2012, 2012]\), for a total of \(\binom{4025}{2013}\) possible functions.