HMMT Monthly Contest
October 2015

Please submit solutions to the following problems at [hmmt.mit.edu/tournaments/monthly/submit] by October 31st, 2015 at 11:59 PM Eastern Standard Time (EST). Before you submit, make sure you have read the rules at [hmmt.mit.edu/tournaments/monthly]. Have fun!

Note: You should not use (or need) any computational aids (calculators, Mathematica, etc.) for the first four problems. For the last problem, however, you are free to use any resources you desire.

1. The function \( f(x) = x^2 - 11x + c \) has roots \( p \) and \( q \), and the function \( g(x) = x^2 - px + q \) has roots \( r \) and \( s \). If the roots of \( h(x) = x^2 - rx + s \) are integers, compute the sum of all possible values of \( c \).

2. For how many pairs \( (m, n) \) of integers with \( 1 \leq m, n \leq 100 \) is it possible to write
   \[
   x^3 + mx^2 + nx + 2
   \]
as the product of two nonconstant polynomials with integer coefficients?

3. Let \( a, b, \) and \( c \) be positive real numbers such that
   \[
   a^2 + ab + b^2 = 67 \\
   b^2 + bc + c^2 = 43 \\
   c^2 + ac + a^2 = 91
   \]
   Find the value of \( ab + 2ac + 3bc \).

4. For all ordered triples of real numbers \((a, b, c)\) such that
   \[
   a + ab + abc = 5 \\
   b + bc + abc = 13 \\
   c + ac + abc = 19
   \]
   let \( s = 19a + 5b + 13c \). Find the sum of all possible values of \( s \).

5. Continuing from problem 1, define the following sequence: let \( f_0(x) = x^2 - ax + b \). For \( n > 0 \), let \( f_n(x) = x^2 - px + q \), where \( p \) and \( q \) are the roots of \( f_{n-1}(x) \) with \( p \geq q \).
   How does \( f_n(x) \) behave as \( n \) approaches infinity? How does this depend on \( a \) and \( b \)? Any interesting results you can prove regarding this sequence? We can’t wait to see what you come up with!