HMMT November 2016

November 12, 2016

General

1. If a and b satisfy the equations $a + \frac{1}{b} = 4$ and $\frac{1}{a} + b = \frac{16}{15}$, determine the product of all possible values of ab.

2. I have five different pairs of socks. Every day for five days, I pick two socks at random without replacement to wear for the day. Find the probability that I wear matching socks on both the third day and the fifth day.

3. Let V be a rectangular prism with integer side lengths. The largest face has area 240 and the smallest face has area 48. A third face has area x, where x is not equal to 48 or 240. What is the sum of all possible values of x?

4. A rectangular pool table has vertices at $(0, 0)$ $(12, 0)$ $(0, 10)$, and $(12, 10)$. There are pockets only in the four corners. A ball is hit from $(0, 0)$ along the line $y = x$ and bounces off several walls before eventually entering a pocket. Find the number of walls that the ball bounces off of before entering a pocket.

5. Let the sequence $\{a_i\}_{i=0}^{\infty}$ be defined by $a_0 = \frac{1}{2}$ and $a_n = 1 + (a_{n-1} - 1)^2$. Find the product
$$\prod_{i=0}^{\infty} a_i = a_0 a_1 a_2 \dots$$

6. The numbers $1, 2 \cdots 11$ are arranged in a line from left to right in a random order. It is observed that the middle number is larger than exactly one number to its left. Find the probability that it is larger than exactly one number to its right.

7. Let ABC be a triangle with $AB = 13$, $BC = 14$, $CA = 15$. The altitude from A intersects BC at D. Let ω_1 and ω_2 be the incircles of ABD and ACD, and let the common external tangent of ω_1 and ω_2 (other than BC) intersect AD at E. Compute the length of AE.

8. Let $S = \{1, 2, \dots 2016\}$, and let f be a randomly chosen bijection from S to itself. Let n be the smallest positive integer such that $f^{(n)}(1) = 1$, where $f^{(i)}(x) = f(f^{(i-1)}(x))$. What is the expected value of n?

9. Let the sequence a_i be defined as $a_{i+1} = 2^{a_i}$. Find the number of integers $1 \le n \le 1000$ such that if $a_0 = n$, then 100 divides $a_{1000} - a_1$.

10. Quadrilateral $ABCD$ satisfies $AB = 8$, $BC = 5$, $CD = 17$, $DA = 10$. Let E be the intersection of AC and BD. Suppose $BE : ED = 1 : 2$. Find the area of $ABCD$.