THE HARVARD-MIT MATHEMATICS TOURNAMENT

ALGEBRA TEST

This test consists of 10 short-answer problems to be solved individually in 50 minutes. Problems will be weighted with point values after the contest based on how many competitors solve each problem. There is no penalty for guessing.

No translators, books, notes, slide rules, calculators, abaci, or other computational aids are permitted other than the official translation sheets. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted.

Our goal is that a closed form answer equivalent to the correct answer will be accepted. However, we do not always have the resourses to determine whether a complicated or strange answer is equivalent to ours. To assist us in awarding you all the points that you deserve, you answers should be simplified as much as possible. Answers must be exact unless otherwise specified.

Correct mathematical notation must be used. No partial credit will be given unless otherwise specified.

If you believe the test contains an error, please submit your protest in writing to Science Center 109 during lunchtime.

Enjoy!

15^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 11 February 2012

Algebra Test

1. Let f be the function such that

$$f(x) = \begin{cases} 2x & \text{if } x \leq \frac{1}{2} \\ 2 - 2x & \text{if } x > \frac{1}{2} \end{cases}$$

What is the total length of the graph of $\underbrace{f(f(\ldots f}_{2012\, f's}(x)\ldots))$ from $x = 0$ to $x = 1$?

2. You are given an unlimited supply of red, blue, and yellow cards to form a hand. Each card has a point value and your score is the sum of the point values of those cards. The point values are as follows: the value of each red card is 1, the value of each blue card is equal to twice the number of red cards, and the value of each yellow card is equal to three times the number of blue cards. What is the maximum score you can get with fifteen cards?

3. Given points a and b in the plane, let $a \oplus b$ be the unique point c such that abc is an equilateral triangle with a, b, c in the clockwise orientation.

   Solve $(x \oplus (0, 0)) \oplus (1, 1) = (1, -1)$ for x.

4. During the weekends, Eli delivers milk in the complex plane. On Saturday, he begins at z and delivers milk to houses located at z^3, z^5, $z^7, \ldots,$ z^{2013}, in that order; on Sunday, he begins at 1 and delivers milk to houses located at z^2, z^4, $z^6, \ldots,$ z^{2012}, in that order. Eli always walks directly (in a straight line) between two houses. If the distance he must travel from his starting point to the last house is $\sqrt{2012}$ on both days, find the real part of z^2.

5. Find all ordered triples (a, b, c) of positive reals that satisfy: $\lfloor a \rfloor bc = 3, a \lfloor b \rfloor c = 4$, and $ab \lfloor c \rfloor = 5$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

6. Let $a_0 = -2$, $b_0 = 1$, and for $n \geq 0$, let

$$a_{n+1} = a_n + b_n + \sqrt{a_n^2 + b_n^2},$$
$$b_{n+1} = a_n + b_n - \sqrt{a_n^2 + b_n^2}.$$

   Find a_{2012}.

7. Let $\otimes$ be a binary operation that takes two positive real numbers and returns a positive real number. Suppose further that $\otimes$ is continuous, commutative ($a \otimes b = b \otimes a$), distributive across multiplication ($a \otimes (bc) = (a \otimes b)(a \otimes c)$), and that $2 \otimes 2 = 4$. Solve the equation $x \otimes y = x$ for y in terms of x for $x > 1$.

8. Let $x_1 = y_1 = x_2 = y_2 = 1$, then for $n \geq 3$ let $x_n = x_{n-1}y_{n-2} + x_{n-2}y_{n-1}$ and $y_n = y_{n-1}y_{n-2} - x_{n-1}x_{n-2}$. What are the last two digits of $|x_{2012}|$?

9. How many real triples (a, b, c) are there such that the polynomial $p(x) = x^4 + ax^3 + bx^2 + ax + c$ has exactly three distinct roots, which are equal to $\tan y$, $\tan 2y$, and $\tan 3y$ for some real y?

10. Suppose that there are 16 variables $\{a_{i,j}\}_{0 \leq i,j \leq 3}$, each of which may be 0 or 1. For how many settings of the variables $a_{i,j}$ do there exist positive reals $c_{i,j}$ such that the polynomial

$$f(x, y) = \sum_{0 \leq i,j \leq 3} a_{i,j} c_{i,j} x^i y^j$$

   $(x, y \in \mathbb{R})$ is bounded below?

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Algebra Test

Name ______________________________ Team ID# ________

School ____________________ Team ____________________

1. ______________
2. ______________
3. ______________
4. ______________
5. ______________
6. ______________
7. ______________
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9. ______________
10. ______________

Score: __________